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THE POPULATION FREQUENCIES OF SPECIES AND THE ESTIMATION OF POPULATION PARAMETERS

BY I. J. GOOD

A random sample is drawn from a population of animals of various species. (The theory may also be applied to studies of literary vocabulary, for example.) If a particular species is represented r times in the sample of size N, then r/N is not a good estimate of the population frequency, p, when r is small. Methods are given for estimating p, assuming virtually nothing about the underlying population. The estimates are expressed in terms of smoothed values of the numbers n_r ($r = 1, 2, 3, \ldots$), where n_r is the number of distinct species that are each represented r times in the sample. (n_r may be described as 'the frequency of the frequency r'.) Turing is acknowledged for the most interesting formula in this part of the work. An estimate of the proportion of the population represented by the species occurring in the sample is an immediate corollary. Estimates are made of measures of heterogeneity of the population, including Yule's 'characteristic' and Shannon's 'entropy'. Methods are then discussed that do depend on assumptions about the underlying population. It is here that most work has been done by other writers. It is pointed out that a hypothesis can give a good fit to the numbers n_r but can give quite the wrong value for Yule's characteristic. An example of this is Fisher's fit to some data of Williams's on Macrolepidoptera.

1. *Introduction.* We imagine a random sample to be drawn from an infinite population of animals of various species. Let the sample size be N and let n_r distinct species be each represented exactly r times in the sample, so that

$$\sum_{r=1}^{\infty} r n_r = N. \tag{1}$$

The sample tells us the values of $n_1, n_2, \ldots$, but not of n_0. In fact it is not quite essential that n_0 should be finite though we shall find it convenient to suppose that it is.

We shall suggest a method of estimating, among other things,

(i) the population frequency of each species;

(ii) the total population frequency of all species represented in the sample, or, as we may say, 'the proportion of the population represented by (the species occurring in) the sample';

(iii) various general population parameters measuring heterogeneity, including 'entropy'. By 'general' parameters we mean parameters defined without reference to any special form of hypothesis. In §7 we shall consider the estimation of parameters for hypotheses of special forms.

Our results are applicable, for example, to studies of literary vocabulary, of accident proneness and of chess openings, but for definiteness we formulate the theory in terms of species of animals.

The formula (2) was first suggested to me, together with an intuitive demonstration, by Dr A. M. Turing several years ago. Hence a very large part of the credit for the present paper should be given to him, and I am most grateful to him for allowing me to publish this work.

Reasonably precise conditions under which our general results are applicable will be given in §4, but we state at once that the larger is n_1 the more applicable the results. When n_1 is large, n_0 will also be large, but we shall not for the most part attempt to estimate it. There will be a fleeting reference to the estimation of n_0 at the end of §5 and a few more references in §§7 and 8. (See, for example, equation (73).) For populations of known finite size, the

problem has been considered by Goodman (1949). He proved that if the sample size is not less than the maximum number of individuals in the population belonging to a single species, then there is only one unbiased estimate of n_0 and he found it. He also pointed out that the unbiased estimate is liable to be unreasonable and suggested some alternative estimates that are always reasonable. There is practically no overlapping between the present work and that of Goodman.

Jeffreys (1948, §3·23) has discussed what is superficially the same problem as (i) above, under the heading 'multiple sampling'. He refers to some earlier work of Johnson (1932). The methods of Johnson and Jeffreys depend on assumptions that, as Jeffreys himself points out, are not always acceptable. Moreover, their methods are not intended to be applicable when n_0 is unknown. The matter is taken up again in §2.

Other work on the frequencies of species has been mainly concerned with the fitting of particular distributions to the data, with or without a theoretical explanation of why these distributions might be expected to be suitable. See, for example, Anscombe (1950), Chambers & Yule (1942), Corbet, Fisher & Williams (1943), Greenwood & Yule (1920), Newbold (1927), Preston (1948), Yule (1944) and Zipf (1932). The methods of the first six sections of the present paper are largely independent of the distributions of population frequencies.

We shall be largely concerned with q_r, the population frequency of an arbitrary species that is represented r times in the sample. We shall use the notation $\mathscr{E}(q_r)$ for the expected value of q_r, in a sense to be explained in §2. Our main result, expressed rather loosely, is that the expected value of q_r is r^*/N, where

$$r^* \simeq (r+1)\, n_{r+1}/n_r. \tag{2}$$

(The symbol '$\simeq$' is used throughout to mean 'is approximately equal to'.) More precisely the n_r's should first be smoothed before applying formula (2). Smoothing is briefly discussed in §3 with examples in §8. If the smoothed values are denoted by $n'_1, n'_2, n'_3, \ldots$, then the more accurate form of equation (2) is

$$r^* \simeq (r+1)\, n'_{r+1}/n'_r. \tag{2'}$$

The reader will find it instructive to consider the special case when n'_r is of the Poisson form $s\, e^{-a} a^r/r!$ Then r^* reduces to a constant.

The formula (2) can be generalized to give higher moments of q_r. In fact

$$\mathscr{E}(q_r^m) \simeq \frac{(r+m)^{(m)}}{N^m} \frac{n_{r+m}}{n_r} \quad (r = 1, 2, 3, \ldots;\ m = 0, 1, 2, \ldots), \tag{3}$$

where $t^{(m)} = t(t-1)\ldots(t-m+1)$. We can also write (3) in the form

$$\mathscr{E}(q_r^m) \simeq \mathscr{E}(q_r)\, \mathscr{E}(q_{r+1}) \ldots \mathscr{E}(q_{r+m-1}). \tag{4}$$

Moreover, the variance of q_r is

$$\begin{aligned} V(q_r) &\simeq \frac{(r+1)(r+2)}{N^2} \frac{n_{r+2}}{n_r} - \left(\frac{r+1}{N} \frac{n_{r+1}}{n_r}\right)^2 \\ &\simeq \mathscr{E}(q_r)\, [\mathscr{E}(q_{r+1}) - \mathscr{E}(q_r)]. \end{aligned} \tag{5}$$

An immediate deduction from (2) is that *the expected total chance of all species that are each represented r times* ($r \geqslant 1$) *in the sample is approximately*

$$(r+1)\, n_{r+1}/N. \tag{6}$$

Hence also the expected total chance of all species that are represented r times or more in the sample is approximately

$$N^{-1}\{(r+1)n_{r+1}+(r+2)n_{r+2}+\ldots\}. \tag{7}$$

In particular, the expected total chance of all species represented at all in the sample is approximately

$$N^{-1}(2n_2+3n_3+\ldots) = 1-n_1/N. \tag{8}$$

We may say that the proportion of the population represented by the sample is approximately $1-n_1/N$, and *the chance that the next animal sampled will belong to a new species is approximately*

$$n_1/N. \tag{9}$$

(Thus (6) is true even if $r = 0$.)

The results (6), (7), (8) and (9) are improved in accuracy by writing the respective formulae as

$$\frac{(r+1)n'_{r+1}n_r}{n'_r N}, \tag{6'}$$

$$N^{-1}\left\{\frac{(r+1)n'_{r+1}n_r}{n'_r}+\frac{(r+2)n'_{r+2}n_{r+1}}{n'_{r+1}}+\ldots\right\}, \tag{7'}$$

$$N^{-1}\left\{\frac{2n'_2 n_1}{n'_1}+\frac{3n'_3 n_2}{n'_2}+\ldots\right\} \tag{8'}$$

and

$$\frac{n'_1 n_0}{n'_0 N}. \tag{9'}$$

In most applications this last expression will be extremely close to n'_1/N, and this in its turn will often be very close to n_1/N. It follows that (8') and (9') are practically the same as (8) and (9). For the sake of mathematical consistency, the smoothing should be such that (8') and (9') add up to 1.

An index of notations used in a fixed sense is given in §9.

I am grateful, and my readers will also be grateful, to Prof. M. G. Kendall for forcing me to clarify some obscurities, especially in §§1 and 2.

2. *Proofs.* Let the number of species in the population be s, which we suppose is finite. This is the same supposition as that n_0 is finite. Our results as far as §6 would be practically unchanged if s were enumerably infinite, but the proofs are more rigorous when it is finite. Let the population frequencies of the species be, in some order, $p_1, p_2, \ldots, p_s$, where

$$p_1+p_2+\ldots+p_s = 1, \quad n_0+n_1+\ldots = s.$$

Let H, or more explicitly $H(p_1, p_2, \ldots, p_s)$, be the statistical hypothesis asserting that $p_1, p_2, \ldots, p_s$ are the population frequencies. We shall discuss the expectation of n_r, given H. It may be objected that the expectation of n_r is simply the observed number n_r, whatever the information, and this objection would be logically correct. Strictly we should introduce extra notation, say $\nu_{r,N}$, for the random variable that is the frequency of the frequency r in a random sample of size N. Then we could introduce the notation $\mathscr{E}(\nu_{r,N} \mid H)$ for the expectation of $\nu_{r,N}$ given H. (Logically this expectation would remain unaffected if particular values of $n_1, n_2, n_3, \ldots$ were given.) In order to avoid the extra notation $\nu_{r,N}$ we shall write $\mathscr{E}(n_r)$ or $\mathscr{E}(n_r \mid H)$ or $\mathscr{E}_N(n_r \mid H)$ instead of $\mathscr{E}(\nu_{r,N} \mid H)$. Confusion can be avoided by reading $\mathscr{E}_N(n_r \mid H)$ as 'the expectation of the frequency of the frequency r when H is given

and when the sample size is N'. Similarly, we write $V(n_r) = V(n_r \mid H) = V_N(n_r \mid H)$ for the variance of $\nu_{r,N}$ given H and $\mathscr{E}_N(n_r^2 \mid H)$, etc., for $\mathscr{E}(\nu_{r,N}^2 \mid H)$.

We recall the theorem that an expectation of a sum is the sum of the expectations. It follows that $\mathscr{E}_N(n_r \mid H)$ is the sum over all s species of the probabilities that each will occur r times, given H. So

$$\mathscr{E}_N(n_r \mid H) = \mathscr{E}(n_r \mid H) = \mathscr{E}(n_r)$$
$$= \sum_{\mu=1}^{s} \binom{N}{r} p_\mu^r (1-p_\mu)^{N-r}. \tag{10}$$

In particular
$$\mathscr{E}_N(n_0 \mid H) = \sum_{\mu=1}^{s} (1-p_\mu)^N. \tag{11}$$

If s were infinite this series would diverge. The divergence would be appropriate since n_0 would also be infinite.

Now suppose that in a sample of size N a particular species occurs r times ($r = 0, 1, 2, \ldots$). We shall consider the final (*posterior*) probability that this species is the μth one (of population frequency p_μ). For the sake of rigour it is necessary to define more precisely how the species is selected for consideration. We shall suppose that it is sampled 'at random', or rather equiprobably, from the s species, and that then its number of occurrences in the sample is counted. Thus the initial (*prior*) probability that the species is the μth one is $1/s$. If the species is the μth one then the likelihood that the observed number of occurrences is r is

$$\binom{N}{r} p_\mu^r (1-p_\mu)^{N-r}.$$

We write q_r for the (unknown) population frequency of an arbitrary species that is represented r times in the sample. The final probability that the species is the μth one can be written as $P(q_r = p_\mu \mid H)$ provided that the p_μ's are unequal. (If any of the p_μ's are equal they can be adjusted microscopically so as to be made unequal. These adjustments will have no practical effect.) We may at once deduce the final probability that the species is the μth one by using Bayes's theorem in the form that the final probabilities are proportional to the initial ones times the likelihoods. We find that

$$P(q_r = p_\mu \mid H) = \frac{p_\mu^r (1-p_\mu)^{N-r}}{\sum_{\mu=1}^{s} p_\mu^r (1-p_\mu)^{N-r}}. \tag{12}$$

It follows that for any positive integer m,

$$\mathscr{E}(q_r^m \mid H) = \frac{\sum_{\mu=1}^{s} p_\mu^{r+m} (1-p_\mu)^{N-r}}{\sum_{\mu=1}^{s} p_\mu^r (1-p_\mu)^{N-r}} \tag{13}$$

$$= \frac{(r+m)^{(m)}}{(N+m)^{(m)}} \frac{\mathscr{E}_{N+m}(n_{r+m} \mid H)}{\mathscr{E}_N(n_r \mid H)}, \tag{14}$$

in view of (10) and of (10) with N replaced by $N+m$. Immediate consequences of (14) are the basic result

$$\mathscr{E}(q_r \mid H) = \frac{r+1}{N+1} \frac{\mathscr{E}_{N+1}(n_{r+1} \mid H)}{\mathscr{E}_N(n_r \mid H)} \quad (r = 0, 1, 2, \ldots) \tag{15}$$

and
$$V(q_r \mid H) = \frac{\mu'_{r,2,N} \mu'_{r,0,N} - \mu'^2_{r,1,N}}{\mu'^2_{r,0,N}}, \tag{16}$$

where $$\mu'_{r,t,N} = \frac{(r+t)!}{(N+t)!}\mathscr{E}_{N+t}(n_{r+t} \mid H) \tag{17}$$

$$= \frac{1}{(N-r)!}\sum_{\mu=1}^{s} p_\mu^{r+t}(1-p_\mu)^{N-r} = \frac{r!}{N!}\mathscr{E}_N(n_r \mid H)\,\mathscr{E}(q_r^t \mid H) \tag{18}$$

by (10) and (14). It is clear from either form of (18) that the numbers $\mu'_{r,t,N}$ $(t = 0, 1, 2, \ldots)$ form a sequence of moment constants and therefore satisfy Liapounoff's inequality. (See, for example, Good (1950*a*), or Uspensky (1937).) This checks that the right side of (16) is positive, as it should be being a variance. [It is obvious incidentally that (16) would be true with $\mu'_{r,t,N}$ defined as $\mathscr{E}(q_r^t \mid H)$ times any expression independent of t.]

We can now approximate the formulae (14) and (15) by replacing $\mathscr{E}_{N+m}(n_{r+m} \mid H)$ by the observed value, n_{r+m}, in the sample of size N, or rather by the smoothed value n'_{r+m}. If m is very small compared with N, if n_r and n_{r+m} are not too small and if the sequence $n_1, n_2, n_3, \ldots$ is smoothed in the neighbourhood of n_r and n_{r+m}, then we may expect the approximations to be good. We thus obtain all the approximate results of §1. Note that when the approximation is made of replacing $\mathscr{E}_{N+m}(n_{r+m} \mid H)$ by n'_{r+m} we naturally also change the notation $\mathscr{E}(q_r^m \mid H)$ to $\mathscr{E}(q_r^m)$. For the results become roughly independent of H unless the n_r's are too small to smooth. Observe that $\mathscr{E}(q_r^m \mid H)$ does not depend on the sample, unless H is itself determined by using the sample. On the other hand, $\mathscr{E}(q_r^m)$ does depend on the sample. This may seem a little paradoxical and the following explanation is perhaps worth giving. When we select a particular sequence of smoothed values $n'_1, n'_2, n'_3, \ldots$ we are virtually accepting a particular hypothesis H, say $H\{N; n'_1, n'_2, n'_3, \ldots\}$, with curly brackets. (I do not think that this hypothesis is usually a simple statistical hypothesis.) Then $\mathscr{E}(q_r^m)$ can be regarded as a shorthand for $\mathscr{E}(q_r^m \mid H\{N; n'_1, n'_2, n'_3, \ldots\})$. (If $H\{\ldots\}$ is not a simple statistical hypothesis this last expression could in theory be given a definite value by assuming a definite distribution of probabilities of the simple statistical hypotheses of which H is a disjunction.) When we regard the smoothing as reasonably reliable we are virtually taking $H\{N; n'_1, n'_2, n'_3, \ldots\}$ for granted, as an approximation, so that it can be omitted from the notation without serious risk of confusion. In order to remind ourselves that there is a logical question that is obscured by the notation, we may describe $\mathscr{E}(q_r^m)$ as say a 'credential expectation'.

If a specific H is accepted it is clearly not necessary to use the approximations since equation (13) can then be used directly. Similarly, if H is assumed to be selected from a superpopulation, with an assigned probability density, then again it is theoretically possible to dispense with the approximations. In fact if the 'point' $(p_1, p_2, \ldots, p_s)$ is assumed to be selected from the 'simplex' $p_1 + p_2 + \ldots + p_s = 1$, with probability density proportional to $(p_1 p_2 \ldots p_s)^{k-1}$, where k is a constant, then it is possible to deduce Johnson's estimate $q_r = (r+k)/(N+ks)$. Jeffreys's estimate is the special case $k = 1$, when the probability density is uniform. Jeffreys suggests conditions for the applicability of his estimate, but these conditions are not valid for our problem in general. This is clear if only because we do not assume s to be known.

Jeffreys assumes explicitly that all ordered partitions of N into s non-negative parts are initially equally probable, while Johnson assumes that the probability that the next individual sampled will be of a particular species depends only on N and on the number of times that that species is already represented in the sample. Clearly both methods ignore any information that can be obtained from the entire set of frequencies of all species.

The ignored information is considerable when it is reasonable to smooth the frequencies of the frequencies.

3. *Smoothing.* The purpose of smoothing the sequence $n_1, n_2, n_3, \ldots$ and replacing it by a new sequence $n_1', n_2', n_3', \ldots$, is to be able to make sensible use of the exact results (14) and (15). Ignoring the discrepancy between $\mathscr{E}_N$ and $\mathscr{E}_{N+m}$, the best value of n_r' would be $\mathscr{E}_N(n_r \mid H)$, where H is true. One method of smoothing would be to assume that $H = H(p_1, p_2, \ldots, p_s)$ belongs to some particular set of possible H's, to determine one of these, say H_0, by maximum likelihood and then to calculate n_r' as $\mathscr{E}_N(n_r \mid H_0)$. This method is closely related to that of Fisher in Corbet *et al.* (1943). Since one of our aims is to suggest methods which are virtually distribution-free, it would be most satisfactory to carry out the above method using all possible H's as the set from which to determine H_0. Unfortunately, this theoretically satisfying method leads to a mathematical problem that I have not solved.

It is worth noticing that the sequence $\{\mathscr{E}_N(n_r \mid H)\}\,(r = 0, 1, 2, \ldots)$ has some properties invariant with respect to changes in H. Ideally the sequence $\{n_r'\}$ should be forced to have these invariant properties. In particular the sequence $\{\mu'_{r,t,N}\}\,(t = 0, 1, 2, \ldots)$, defined by (17), is a sequence of moment constants. But if $t = o(\sqrt{N})$, then $N^{-t}(r+t)!\, n'_{r+t} \simeq \mu'_{r,t,N}$, so that if $t = o(\sqrt{N})$ we can assume that the sequence $r!\, n_r'$ is a sequence of moment constants and satisfies Liapounoff's inequalities. But this simply implies that $0^*, 1^*, 2^*, \ldots, t^*$ forms an increasing sequence (see equation (2')), a result which is intuitively obvious even without the restriction $t = o(\sqrt{N})$. (Indeed, the argument could be reversed in order to obtain a new proof of Liapounoff's inequality.) We also intuitively require that $0^*, 1^*, 2^* \ldots$ should itself be a 'smooth' sequence.

Since the sequence $\{\mu'_{r,t,N}\}\,(t = 0, 1, 2, \ldots)$ is a sequence of moment constants of a probability distribution it follows from Hardy (1949, §11·8) that the sequence is 'totally increasing', i.e. that all its finite differences are non-negative. This result is unfortunately too weak to be useful for our purposes, but it may be possible to make use of some other theorems concerning moment constants. This line of research will not be pursued in the present paper.

A natural principle to adopt when smoothing is that

$$\chi^2 = \sum_{i=1}^{r} \frac{(n_i' - n_i)^2}{V(n_i \mid H)} \tag{19}$$

should not be significant with r degrees of freedom. In §5 we shall obtain an approximate formula for $V(n_r \mid H)$, applicable when $r^2 = o(N)$. The chi-squared test will therefore be applicable when $r^2 = o(N)$. [See formulae (22), (25), (26) and, for particular H's, (65), (85), (86).]

Another similar principle can be understood by thinking of the histogram of n_r as several piles of pennies, n_r pennies in the rth pile. We may visualize the smoothing as the moving of pennies from pile to pile, and we may reasonably insist that pennies moved to the rth pile should not have been moved much further horizontally than a distance $\sqrt{r}$ and almost never further than $2\sqrt{r}$. For $r = 0$ we would not insist on this rule, i.e. we do not insist that $\sum_{r=1}^{\infty} n_r' = \sum_{r=1}^{\infty} n_r$. The analogy with piles of pennies amounts to saying that a species that 'should' have occurred r times is unlikely to have occurred less than $r - \sqrt{r}$ or more than $r + \sqrt{r}$ times.

Let $N' = \Sigma r n'_r$. It seems unnecessary to insist on $N' = N$, provided that N is replaced by N' in such formulae as $\mathscr{E}(q_r) \doteq r^*/N$. It will be convenient, however, in § 6 to assume $N' = N$.

For some applications very little smoothing will be required, while for others it may be necessary to use quite elaborate methods. For example, we could

(i) Smooth the n_r's for the range of values of r that interests us, holding in mind the above chi-squared test and the rule concerning $\sqrt{r}$. The smoothing techniques may include the use of freehand curves. Rather than working directly with $n_1, n_2, n_3, \ldots$ it may be found more suitable to work with the cumulative sums $n_1, n_1+n_2, n_1+n_2+n_3, \ldots$ or with the cumulative sums of the rn_r or with the logarithms $\log n_1$, $\log n_2$, $\log n_3, \ldots$. There is much to be said for working with the numbers $\sqrt{n_1}, \sqrt{n_2}, \sqrt{n_3}, \ldots$. For if we assume that $V(n_r \mid H)$ is approximately equal to n_r (and in view of (26) and (27) of § 3 this approximation is not on the whole too bad), then it would follow that the standard deviation of $\sqrt{n_r}$ is of the order of $\frac{1}{2}$ and therefore largely independent of r. Hence graphical and other smoothing methods can be carried out without having constantly to hold in mind that $|n'_r - n_r|$ can reasonably take much larger values when n_r is large than when it is small. [The square-root transformation for a Poisson variable, x, was suggested by Bartlett (1936) in order to facilitate the analysis of variance. He showed also that the transformation $\sqrt{(x+\frac{1}{2})}$ leads to an even more constant variance. Anscombe (1948) proved that $\sqrt{(x+\frac{3}{8})}$ has the most nearly constant variance of any variable of the form $\sqrt{(x+c)}$, namely, $\frac{1}{4}$, when the mean of x is large. He attributes this result to A. H. L. Johnson.]

(ii) Calculate $(r+1)\, n'_{r+1}/n'_r$.

(iii) Smooth these values getting, say, r^*.

(iv) Possibly use the values of r^* to improve the smoothing of the n_r's. If this makes a serious difference it will be necessary to check again that the chi-squared test and the $\sqrt{r}$ rule have not been violated.

(v) Light can be shed on the reliability of the estimates of the q_r's, etc., if the data are smoothed two or three times, possibly by different people.

In short, the estimation of the q_r's should be done in such a way as to be consistent with the axioms of probability and also with any intuitive judgements that the users of the method are not prepared to abandon or to modify. (This recommendation applies to much more general theoretical scientific work, though there are rare occasions when it may be preferred to abandon the axioms of a science.)

An objection could be raised to the methods of smoothing suggested in the present section. It could be argued that all smoothing methods indirectly assume something about the distribution p_μ, and that one might just as well apply the method of Greenwood & Yule (1920) and its modification by Corbet *et al.* (1943) of assuming a distribution of Pearson's Type III, $Ap^\alpha e^{-\beta p}$, or of some other form. Our reply would be that smoothing can be done by making only *local* assumptions, for example, that the square root of $\mathscr{E}(n_r \mid H)$, as a function of r, is approximately 'parabolic' for any nine consecutive values of r. Moreover, it may often be more convenient to apply the general methods of the present section than to attempt to find an adequate hypothesis, H.

4. *Conditions for the applicability of the results of* §§ 1 *and* 2. The condition for the applicability of the results of §§ 1 and 2 is that the user of the methods should be satisfied with his approximations to $\mathscr{E}_{N+m}(n_{r+m} \mid H)$ corresponding to the values of r and m used in the application. This condition is clearly correct, since equation (14) is exact. In particular, if

n_1 is large enough the user would be quite happy to deduce (9) from (15) with $r = 0$. Similarly, he will be satisfied with the estimates of say q_1, q_2 and q_3 provided he is satisfied with the smoothed values (n_1', n_2', n_3', n_4') of n_1, n_2, n_3 and n_4.

5. *The variance of* n_r. For the application of the chi-squared test described in §3 we need to know more about $V(n_r)$. We begin by obtaining an exact formula for $V(n_r \mid H) = V_N(n_r \mid H)$ and we then make approximations that justify the omission of the symbol H from the notation. It is convenient to introduce the random variable $x_{\mu,r} = x_\mu$ that is defined as 1 if the 'μth species' (of population frequency p_μ) occurs precisely r times in a sample of size N (H being given), otherwise $x_\mu = 0$. Clearly $P(x_\mu = 1 \mid H) = \binom{N}{r} p_\mu^r (1-p_\mu)^{N-r}$. Now

$$\begin{aligned}
\mathscr{E}(n_r^2 \mid H) &= \mathscr{E}(\sum_\mu x_\mu)^2 \\
&= \sum_{\mu,\nu} \mathscr{E}(x_\mu x_\nu) \\
&= \sum_\mu \mathscr{E}(x_\mu) + \sum_{\mu,\nu}^{\mu+\nu} \mathscr{E}(x_\mu x_\nu) \\
&= \mathscr{E}(n_r \mid H) + \frac{N!}{r!\,r!\,(N-2r)!} \sum_{\mu,\nu}^{\mu+\nu} p_\mu^r p_\nu^r (1-p_\mu-p_\nu)^{N-2r}. \qquad (20)
\end{aligned}$$

This is exact. We now make some approximations of the sort used in deriving the Poisson distribution from the binomial. We get, assuming r^2/N, rp_μ and rp_ν to be small,

$$\binom{N}{r} p_\mu^r (1-p_\mu)^{N-r} \simeq \frac{(Np_\mu)^r e^{-Np_\mu}}{r!} = a_\mu, \text{ say,}$$

and

$$\frac{N!}{r!\,r!\,(N-2r)!} p_\mu^r p_\nu^r (1-p_\mu-p_\nu)^{N-2r} \simeq a_\mu a_\nu.$$

Moreover, it is intuitively clear that terms for which p_μ or p_ν is far from r/N can make no serious contribution to the summation in (20). Hence, if $r^2 = o(N)$,

$$\begin{aligned}
\mathscr{E}(n_r^2 \mid H) &\simeq \mathscr{E}(n_r \mid H) + \sum_{\mu,\nu}^{\mu+\nu} a_\mu a_\nu \\
&\simeq \mathscr{E}(n_r \mid H) + \{\mathscr{E}(n_r \mid H)\}^2 - \sum_\mu a_\mu^2.
\end{aligned}$$

Therefore the variance of n_r for samples of size N is

$$V_N(n_r \mid H) \simeq \mathscr{E}_N(n_r \mid H) - \sum_\mu \frac{(Np_\mu)^{2r} e^{-2Np_\mu}}{(r!)^2} \qquad (21)$$

$$\simeq \mathscr{E}_N(n_r \mid H) - \frac{1}{2^{2r}} \binom{2r}{r} \mathscr{E}_{2N}(n_{2r} \mid H). \qquad (22)$$

Formulae (21) and (22) are elegant but need further transformation, when H is unknown, before they can be used for calculation. Notice first that there are n_u species whose expected population frequencies are q_u $(u = 0, 1, 2, \ldots)$. Hence we have for $r = 0, 1, 2, \ldots$; $r^2 = o(N)$,

$$\begin{aligned}
V(n_r \mid H) &\simeq \mathscr{E}(n_r \mid H) - \sum_{u=0}^{\infty} n_u \left\{ \frac{(Nq_u)^r e^{-Nq_u}}{r!} \right\}^2 \\
&= \mathscr{E}(n_r \mid H) - \sum_{u=0}^{\infty} n_u \left\{ \frac{u^{*r} e^{-u^*}}{r!} \right\}^2. \qquad (23)
\end{aligned}$$

Similarly and rather more simply, when $r^2 = o(N)$,

$$\mathscr{E}(n_r \mid H) \simeq \frac{1}{r!} \sum_{u=0}^{\infty} n_u u^{*r} e^{-u^*}. \tag{24}$$

Now for any positive x, $x^r e^{-x} \leqslant r^r e^{-r}$, so

$$\mathscr{E}(n_r \mid H) \gtrsim V(n_r \mid H) \gtrsim \mathscr{E}(n_r \mid H)\left(1 - \frac{r^r e^{-r}}{r!}\right). \tag{25}$$

Using Stirling's formula for $r-1$ we have

$$\mathscr{E}(n_r \mid H) \gtrsim V(n_r \mid H) \gtrsim \mathscr{E}(n_r \mid H)\left(1 - \frac{1}{\sqrt{(2\pi r)}}\right) \quad (r = 2, 3, \ldots), \tag{26}$$

while

$$\mathscr{E}(n_1 \mid H) \gtrsim V(n_1 \mid H) \gtrsim \mathscr{E}(n_1 \mid H)(1 - e^{-1}) \tag{27}$$

(see also formula (65) in §7). Now the most desirable value for n'_r would be $\mathscr{E}(n_r \mid H)$ where H is true, so if our smoothing of the n_r's is to be satisfactory for any particular values of r small compared with $\sqrt{N}$ we may write

$$n'_r \simeq \sum_{u=0}^{\infty} n_u \frac{u^{*r} e^{-u^*}}{r!}, \tag{28}$$

and these approximate equations may be used as a test of consistency for the values of n'_r and u^*. Indeed, it may be possible iteratively to solve equations (28) combined with (2′) and thus very systematically to obtain estimates of n'_r and r^* for values of r small compared with $\sqrt{N}$. This iterative process may possibly lead to estimates of n'_0 and 0^*, but I have not yet tried out the process. For most applications the less systematic methods previously described will probably prove to be adequate, and any smoothing obtained by these methods can be partially tested by means of χ^2 in the form (19), together with the inequalities (26) and (27). (See also the remarks following equations (65) and (87).)

6. *Estimation of some population parameters, including entropy.* Let us consider the population parameters

$$c_{m,n} = \sum_{\mu=1}^{s} p_\mu^m (-\log p_\mu)^n \quad (m, n = 0, 1, 2, \ldots), \tag{29}$$

which can be regarded as measures of heterogeneity of the population. The sequence $c_{0,0} = 1, c_{1,0} = s, c_{2,0}, c_{3,0}, \ldots$ may be called the 'moment constants' of the population, while $c_{1,1}$ is called the 'entropy' in the modern theory of communication (see Shannon, 1948). More generally, $c_{1,n}$ is the moment about zero of the amount of information from each selection of an animal (or word), where 'amount of information' is here used in the sense of Good (1950*b*, p. 75), i.e. as minus the logarithm of a probability. (The last sentence of p. 75 of this reference is incorrect, as Prof. M. S. Bartlett has pointed out.) We find it no more difficult to give estimates of $c_{m,n}$ than of $c_{1,n}$, at any rate when $n = 0$ or 1.

It is an immediate consequence of (10) that an unbiased estimate of $c_{m,0}$ is

$$\hat{c}_{m,0} = \frac{1}{N^{(m)}} \sum_r r^{(m)} n_r. \tag{30}$$

$\hat{c}_{2,0}$ is in effect used by Yule (1944) to measure the heterogeneity of samples of vocabulary, and he calls $10{,}000\hat{c}_{2,0}(1 - 1/N)$ the 'characteristic' of the material. The sequence of all sampling moments of $\hat{c}_{2,0}$ involves all the population parameters $c_{m,0}$. For example, as pointed out by Simpson (1949), for large N,

$$V(\hat{c}_{2,0}) \simeq \frac{4}{N}(c_{3,0} - c_{2,0}^2). \tag{30 A}$$

Unbiased statistics are rather unfashionable nowadays, partly because they can take impossible values. For example, $\hat{c}_{m,0}$ could vanish, although it is easy to see that $c_{m,0} \geqslant s^{-(m-1)}$. (Compare Good (1950*b*, p. 103), where estimates of $c_{m,0}$ are implicit for general multinomial distributions, no attempt being made to smooth the n_r's.) We shall find estimates of $c_{m,1}$ and also estimates of $c_{m,0}$ that are at least sometimes better than $\hat{c}_{m,0}$.

We have

$$c_{m,0} = \frac{1}{N^{(m)}} \sum_r r^{(m)} \mathscr{E}(n_r \mid H), \tag{31}$$

since this is in effect what is meant by saying that $\hat{c}_{m,0}$ is an unbiased estimate of $c_{m,0}$. If the statistician is satisfied with his smoothing, i.e. if he assumes that $n'_r \simeq \mathscr{E}(n_r \mid H)$, and if he has forced $N' = N$, then he can estimate $c_{m,0}$ as

$$\tilde{c}_{m,0} = \frac{1}{N^{(m)}} \sum_r r^{(m)} n'_r, \tag{32}$$

and he will be prepared to assume that this is a more efficient estimate than $\hat{c}_{m,0}$. More generally if the smoothing is satisfactory for $r = 1, 2, \ldots, t$ but not for all larger values of r, then a good estimate of $c_{m,0}$ will be $\tilde{c}_{m,0}(t)$, where

$$\tilde{c}_{m,0}(t) = \frac{1}{N^{(m)}} \left\{ \sum_{r=m}^{t} r^{(m)} n'_r + \sum_{r=t+1}^{\infty} r^{(m)} n_r \right\}. \tag{33}$$

We shall next consider estimates $\tilde{c}_{m,1}$ of $c_{m,1}$. We shall begin by proving that (exactly)

$$c_{m,1} = \frac{1}{N^{(m)}} \sum_r r^{(m)} \mathscr{E}(n_r \mid H) \left\{ \frac{1}{r+1} + \frac{1}{r+2} + \ldots + \frac{1}{N-r} - \frac{d}{dr} \log \mathscr{E}(n_r \mid H) - \mathscr{E}[\log(1-q_r) \mid H] \right\}. \tag{34}$$

The differential coefficient in this expression is made meaningful by means of a suitable definition of $\mathscr{E}(n_r \mid H)$ for non-integral values of r. This definition is obtained from equation (10) by writing $\Gamma(N+1)/\Gamma(r+1)\,\Gamma(N-r+1)$ instead of $\binom{N}{r}$.

In order to prove (34) we shall need the following generalization of (13), valid for any function $f(.)$:

$$\mathscr{E}(n_r \mid H)\, \mathscr{E}[f(q_r) \mid H] = \binom{N}{r} \sum_\mu p_\mu^r (1-p_\mu)^{N-r} f(p_\mu). \tag{35}$$

We also require the following property of the gamma function. If b is a non-negative integer,

$$\frac{\Gamma'(b+1)}{\Gamma(b+1)} = 1 + \frac{1}{2} + \ldots + \frac{1}{b} - \gamma, \tag{36}$$

where $\gamma = 0{\cdot}577215\ldots$ is the Euler-Mascheroni constant. (See, for example, Jeffreys & Jeffreys (1946, §15·04).) It follows from (10) and (36) that

$$\frac{d}{dr} \mathscr{E}(n_r \mid H) = \binom{N}{r} \sum_\mu p_\mu^r (1-p_\mu)^{N-r} \left(\frac{1}{r+1} + \frac{1}{r+2} + \ldots + \frac{1}{N-r} + \log \frac{p_\mu}{1-p_\mu} \right)$$

$$= \mathscr{E}(n_r \mid H) \left\{ \frac{1}{r+1} + \ldots + \frac{1}{N-r} + \mathscr{E}\left(\log \frac{q_r}{1-q_r} \mid H \right) \right\},$$

by (35). Therefore

$$\mathscr{E}(n_r \mid H)\left(\frac{1}{r+1}+\ldots+\frac{1}{N-r}\right)-\frac{d}{dr}\mathscr{E}(n_r \mid H)-\mathscr{E}(n_r \mid H)\,\mathscr{E}[\log(1-q_r) \mid H]$$
$$=\mathscr{E}(n_r \mid H)\,\mathscr{E}(-\log q_r \mid H)=\binom{N}{r}\sum_\mu p_\mu^r(1-p_\mu)^{N-r}(-\log p_\mu),$$

by (35) again. Multiplying by $r^{(m)}/N^{(m)}$ and summing with respect to r, we find that the right-hand side of (34) equals

$$\sum_\mu\sum_r\binom{N-m}{r-m}p_\mu^{r-m}(1-p_\mu)^{N-m-(r-m)}p_\mu^m(-\log p_\mu)=c_{m,1}$$

as asserted.

$c_{m,2}$ can be evaluated in a similar manner by first writing down $\left(\frac{d}{dr}\right)^2\mathscr{E}(n_r \mid H)$, but the result is complicated and will be omitted.

As in the estimation of $c_{m,0}$, if the statistician is satisfied with his smoothing, then he can write

$$c_{m,1}\doteq\frac{1}{N^{(m)}}\sum_r r^{(m)}n_r'\left\{\frac{1}{r+1}+\frac{1}{r+2}+\ldots+\frac{1}{N-r}-\frac{d}{dr}\log n_r'-\mathscr{E}[\log(1-q_r) \mid H]\right\}.$$

If N is large the approximation can be written

$$c_{m,1}\doteq\frac{1}{N^{(m)}}\sum_r r^{(m)}n_r'\left\{\log N-\left(1+\frac{1}{2}+\ldots+\frac{1}{r}-\gamma\right)-\frac{d}{dr}\log n_r'-\mathscr{E}\left(\log\frac{1-q_r}{1-\frac{r}{N}} \mid H\right)\right\}.$$

Now it is intuitively clear that $\mathscr{E}\left(q_r-\frac{r}{N}\right)$, which equals $\frac{r^*-r}{N}$, must be $O\left(\frac{\sqrt{r}}{N}\right)=o(N)$, and therefore

$$c_{m,1}\doteq\frac{1}{N^{(m)}}\sum_r r^{(m)}n_r'\left\{\log N-\left(1+\frac{1}{2}+\ldots+\frac{1}{r}-\gamma\right)-\frac{d}{dr}\log n_r'\right\}$$
$$=\tilde{c}_{m,0}\log N-\frac{1}{N^{(m)}}\sum_r r^{(m)}n_r'\left(g_r+\frac{d}{dr}\log n_r'\right), \tag{37}$$

where
$$g_r=1+\frac{1}{2}+\ldots+\frac{1}{r}-\gamma. \tag{38}$$

In particular, the entropy $c_{1,1}\doteq\tilde{c}_{1,1}$, where

$$\tilde{c}_{1,1}=\log N-\frac{1}{N}\sum_r rn_r'\left(g_r+\frac{d}{dr}\log n_r'\right). \tag{39}$$

The differentiation can be performed graphically for all r or by numerical differentiation for $r=3,4,5,\ldots$. (For numerical differentiation see, for example, Jeffreys & Jeffreys (1946, §9·07).) Another estimate of the entropy is $\hat{\tilde{c}}_{1,1}$, where

$$\hat{\tilde{c}}_{1,1}=\log N-\frac{1}{N}\sum_r rn_r\left(g_r+\frac{d}{dr}\log n_r'\right), \tag{40}$$

in which the 'prime' has been omitted from the first occurrence of n_r' in (39). This estimate, $\hat{\tilde{c}}_{1,1}$, has leanings towards being an unbiased estimate of the entropy. It can hardly be as good as (39) when the smoothing is reliable. Perhaps the best method of using the present theory for estimating $c_{m,1}$ is to use the compromise $\tilde{c}_{m,1}(t)$ defined in the obvious way by

analogy with (33). For large values of r, the factor $g_r + \frac{d}{dr}\log n'_r$ may be replaced by $\log r$ to a good approximation. Terms of $\tilde{c}_{m,1}(t)$ for which this approximation is made, i.e. terms of the form $rn_r \log r$ may be regarded as crude and unadjusted.

7. *Special hypotheses, H.* In this section we shall consider some special classes of hypotheses, H, which determine the distribution p_μ. So far we have taken this distribution as discrete for the sake of logical simplicity. In the present section we shall find it convenient to assume that there is a density function, $f(p)$, where $f(p)\,dp$ is the number of species whose population frequencies lie between p and $p+dp$. (The formulae may of course be generalized to arbitrary distributions by using the Stieltjes integral.) Clearly

$$\int_0^1 f(p)\,dp = s, \tag{41}$$

$$\int_0^1 pf(p)\,dp = 1. \tag{42}$$

The expected value of p for an animal at random from the population is

$$\mathscr{E}(p \mid H) = \int_0^1 p^2 f(p)\,dp = c_{2,0}. \tag{43}$$

The appropriate modifications of the previous formulae are obvious. For example, instead of (10) and (20) we have

$$\mathscr{E}_N(n_r \mid H) = \binom{N}{r}\int_0^1 p^r(1-p)^{N-r} f(p)\,dp, \tag{44}$$

$$\mathscr{E}_N(n_r^2 \mid H) = \mathscr{E}_N(n_r \mid H) + \binom{N}{r,r}\int_0^1\int_0^1 p^r q^r (1-p-q)^{N-2r} f(p)f(q)\,dp\,dq - \binom{N}{r,r}\int_0^1 p^{2r}(1-2p)^{N-2r} f(p)\,dp. \tag{45}$$

Notice the elegant checks of (44) and (45) that $\mathscr{E}_0(n_0 \mid H) = s$, $\mathscr{E}_1(n_1 \mid H) = 1$, $V_0(n_0 \mid H) = 0$, $V_1(n_1 \mid H) = 0$. Formula (44) leads to the less precise but often more convenient formula

$$\mathscr{E}_N(n_r \mid H) = \frac{1}{r!}\left[1 + O\left(\frac{r^2}{N}\right)\right]\int_0^1 (pN)^r e^{-pN} f(p)\,dp$$

$$\doteq \frac{1}{r!}\int_0^1 (pN)^r e^{-pN} f(p)\,dp \quad (r^2 = o(N)), \tag{46}$$

while a similar treatment of formula (45) leads back merely to formula (22).

We shall now list a number of different types of possible hypotheses and then discuss them. The normalizing constants are all deduced from (42).

H_1 (Pearson's Type I):

$$f(p) = \frac{(\alpha+\beta+2)!}{(\alpha+1)!\,\beta!}\,p^\alpha(1-p)^\beta \quad (\alpha > -1, \beta > -1). \tag{47}$$

H_2 (Pearson's Type III):

$$f(p) = \frac{\beta^{\alpha+2}}{(\alpha+1)!}\,p^\alpha e^{-\beta p} \quad (\alpha > -1, \beta > 0). \tag{48}$$

H_3 (same as H_2 but with $\alpha = -1$):

$$f(p) = \beta p^{-1} e^{-\beta p} \quad (\beta > 0). \tag{49}$$

H_4: $$f(p) = sAp^{\alpha} \exp(-\beta p - \epsilon p^{-1}) \quad (\beta > 0, \epsilon > 0). \tag{50}$$

H_5 (truncated form of H_3): $$f(p) = \begin{cases} \beta p^{-1} e^{-\beta p} & (p > p_0), \\ 0 & (p < p_0). \end{cases} \tag{51}$$

H_6 (truncated form of another special case of H_2):

$$f(p) = \begin{cases} \dfrac{p^{-2} e^{-\beta p}}{E(p_0 \beta)} & (p > p_0), \\ 0 & (p < p_0), \end{cases} \tag{52}$$

where $E(w) = -\operatorname{Ei}(-w) = \int_w^{\infty} u^{-1} e^{-u} du$. $\operatorname{Ei}(w)$ is known as the 'exponential integral' and has been tabulated several times. (For a list of these tables see Fletcher, Miller & Rosenhead (1946, §§13·2 and 13·21).)

We list also a few less completely formulated hypotheses, H_7, H_8 and H_9, for which the population is not explicitly specified, but only the values of $\mathscr{E}_N(n_r \mid H)$. Hence for these hypotheses the parameters may depend on N.

H_7 (Zipf laws): $$\mathscr{E}(n_r \mid H_7) \propto r^{-\zeta} \quad (r \geqslant 1, \zeta > 0), \tag{53}$$

where ζ is often taken as 2 by Zipf. (See also (94) below.)

H_8 (H_7 with a convergence factor):

$$\mathscr{E}(n_r \mid H_8) = \frac{\lambda x^r}{r^{\zeta}} \quad (r \geqslant 1, \zeta > 0, 0 < x < 1). \tag{54}$$

H_9 (a modification of a special case of H_8):

$$\mathscr{E}(n_r \mid H_9) = \frac{\lambda x^r}{r(r+1)} \quad (r \geqslant 1). \tag{55}$$

We now discuss the nine hypotheses.

(i) H_1 has the advantage that the exact formula (44) can be evaluated in elementary terms. We can see from (41) and (43) that

$$s = \frac{\alpha + \beta + 2}{\alpha + 1}, \tag{56}$$

$$\mathscr{E}(p \mid H_1) = \frac{\alpha + 2}{\alpha + \beta + 3} = \frac{(\alpha + 2)(\alpha + \beta + 2)}{(\alpha + 1)(\alpha + \beta + 3)} \frac{1}{s}. \tag{57}$$

In most applications we want $f(p)$ to be small when p is not small and $\mathscr{E}(p \mid H)$ to be large compared with $1/s$. Hence if a hypothesis of the form H_1 is to be appropriate at all, we shall usually want β to be large, by (47), and α to be close to -1, by (57).

By (44) we see that

$$\mathscr{E}_N(n_r \mid H_1) = \binom{N}{r} \frac{(\alpha + \beta + 2)!\,(\alpha + r)!\,(\beta + N - r)!}{(\alpha + 1)!\,\beta!\,(\alpha + \beta + N + 1)!}. \tag{58}$$

Hence, by (2′), if the smoothed values n_r' and n_{r+1}' were equal to their expectations, given H_1, we would have

$$r^* = \frac{(\alpha + r + 1)(N - r)}{\beta + N - r}. \tag{59}$$

(ii) H_2 can be regarded as a convenient approximation to H_1 if $\beta > 0$. Strictly, the hypothesis H_2 is impossible since it allows values of p greater than 1, but it gives all such

values of p combined a very small probability provided that β is large. H_2 was used by Green wood & Yule (1920) and by Fisher (see Corbet *et al.* 1943). We have

$$s = \frac{\beta}{\alpha+1}, \quad \mathscr{E}(p \mid H_2) = \frac{\alpha+2}{\beta} = \frac{\alpha+2}{\alpha+1}\frac{1}{s}, \tag{60}$$

so that α must be close to -1. Hence, if $r^2 = o(N)$,

$$\mathscr{E}_N(n_r \mid H_2) \simeq \frac{\beta(\alpha+r)!}{(\alpha+1)!\,r!}\left(\frac{N}{N+\beta}\right)^r \left(\frac{\beta}{N}\right)^{\alpha+1}, \tag{61}$$

which is of the negative binomial form.

(iii) Of all hypotheses of the form H_2, Fisher (Corbet *et al.* 1943) was mainly concerned with H_3, the case $\alpha = -1$. (See example (i) in §8 below.) Then

$$s = \infty, \quad \mathscr{E}(p \mid H_3) = \frac{1}{\beta}, \tag{62}$$

$$\mathscr{E}_N(n_r \mid H_3) \simeq \frac{\beta}{r}\left(\frac{N}{N+\beta}\right)^r = \frac{\beta x^r}{r} \quad (r^2 = o(N)), \tag{63}$$

say. For large samples, x (which, unlike β, depends on N) is close to 1 and the factor x^r may be regarded as a convergence factor which prevents $\sum_{r=1}^{\infty} \mathscr{E}_N(n_r \mid H_3)$ from becoming infinite. The convergence factor also increases the likelihood of being able to find a satisfactory fit to given frequencies, n_r, merely because it involves a new parameter.

We see from (22) that

$$V_N(n_r \mid H_3) \simeq \mathscr{E}_N(n_r \mid H_3)\left\{1 - \frac{1}{2^{2r+1}}\binom{2r}{r}\left(\frac{N}{N+\beta}\right)^r\right\}. \tag{64}$$

If $\beta r = o(N)$ it follows that

$$\frac{V_N(n_r \mid H_3)}{\mathscr{E}_N(n_r \mid H_3)} \simeq 1 - \frac{1}{2^{2r+1}}\binom{2r}{r}. \tag{65}$$

Thus in these circumstances $V_N(n_r \mid H_3)$ lies between the bounds given by (26) and (27), being for each r about twice as close to the smaller bound than to the larger one. When applying the chi-squared test, where χ^2 is defined by equation (19), we can hardly go far wrong by assuming (65) to be applicable whatever the distribution determined by H may be. But, of course, we may often be able to improve on (65) when H is specified in terms of the distribution of p. For convenience in applying (65) we give a short table of values of

$$k_r = \left[1 - \frac{1}{2^{r+1}}\binom{2r}{r}\right]^{-1} \tag{66}$$

r	k_r	r	k_r	r	k_r
1	1·33	6	1·13	11	1·10
2	1·23	7	1·12	12	1·09
3	1·19	8	1·11	13	1·09
4	1·16	9	1·11	14	1·08
5	1·14	10	1·10	15	1·08

For larger values of r, the approximation $1 + 1/\{(2\sqrt{(\pi r)})\}$ is correct to two places of decimals.

Suppose we are given a sample of size N and we wish to estimate β and x. The method used by Fisher was to equate the observed values of $\Sigma r n_r = N$ and $\Sigma n_r = S$ to their expected values. (Note that S is the observed number of species and should not be confused with s.) This led him to the equations

$$-\beta \log_e(1-x) = S, \quad N = \beta x/(1-x), \tag{67}$$

$$\frac{N}{S} = \frac{x}{-(1-x)\log_e(1-x)}, \tag{68}$$

which he solved by using a table of $x/(1-x)$ in terms of $\log_{10}(N/S)$.

A theoretically more satisfactory method of estimating β and x would be by minimizing χ^2, defined by (19), with $r = \infty$. This method leads to equations which would be most laborious to solve by hand but which will be given here since large-scale computers now exist. To prevent misunderstanding we mention at once that Fisher obtained a perfectly good fit by the simpler method, in his example, i.e. example (i) of §8 below, though, as pointed out in §8, H_3 must not be too literally regarded as true.

By (65) we may write

$$\chi^2 = \sum_{r=1}^{\infty} k_r \left(\frac{\beta x^r}{r} - 2n_r + \frac{r n_r^2}{\beta x^r} \right). \tag{69}$$

The equations giving β and x will then be

$$\beta^2 \Sigma k_r x^r / r = \Sigma r k_r n_r^2 x^{-r}, \tag{70}$$

$$\beta^2 \Sigma k_r x^r = \Sigma r^2 k_r n_r x^{-r}, \tag{71}$$

and these equations could be solved iteratively.

When β and x are specified the cumulative sums of $\mathscr{E}_N(n_r \mid H_3)$ can be found by making use of the approximation

$$\sum_{t}^{t \geqslant r} \frac{x^t}{t} \simeq E(-r \log_e x) + \frac{x^r}{2r}\left(1 + \tfrac{1}{6}\log_e x - \frac{1}{6r}\right), \tag{72}$$

which will be a very good approximation if the terms involving $\frac{1}{6}\log x$ and $\frac{1}{6r}$ are negligible. This approximation can be obtained by means of the Euler-Maclaurin summation formula. (See, for example, Whittaker & Watson (1935, §7·21).)

(iv) We have just seen that when $\alpha = -1$ in H_2 we obtain $s = \infty$ and of course $\mathscr{E}_N(n_0 \mid H) = \infty$. There are strong indications in examples (ii), (iii) and (iv) of §8 that we may wish to take $\alpha < -1$, and then even worse divergencies occur. For example, if $\alpha = -2$ we would obtain, from (61), the intolerable result

$$\mathscr{E}_N(n_1 \mid H_2)/\mathscr{E}_N(n_2 \mid H_2) = \infty.$$

In order to avoid these divergencies we could in theory use hypothesis H_4, with a small value of ϵ. Unfortunately, this hypothesis seems to be analytically unwieldy; it is mentioned partly for its interest as intermediate between Pearson's Types III and V.

(v) Another method of avoiding divergencies is to use truncated distributions. These truncated distributions are not theoretically pleasing but at least some of them can be handled analytically. H_5 is a truncated form of H_3. We may describe p_0 as the smallest possible population frequency of any species. In most applications it would be difficult to obtain a sample large enough to determine p_0 with any accuracy. In fact if the estimate of

p_0 were to be reliable the sample would need to be so large that n_r would vanish for all small values of r. In the examples of §8, n_1 is always larger than any other value of n_r, so these samples would need to be increased greatly before one could expect even n_1 to vanish.

We obtain from (41)

$$s = \beta E(p_0\beta). \tag{73}$$

Now

$$E(w) = -\gamma - \log_e w + w - \frac{w^2}{2!\,2} + \frac{w^3}{3!\,3} - \ldots, \tag{74}$$

an equation which is undoubtedly well known. It can be proved, for example, by using Dirichlet's formula for γ. (See, for example, Whittaker & Watson (1935, §12·3, example 2).) In particular, if w is very small,

$$E(w) \sim -\log_e(\gamma' w), \tag{75}$$

where

$$\gamma' = e^{\gamma} = 1{\cdot}781072. \tag{76}$$

(Cf. Jahnke & Emde (1933, p. 79), where our γ' is denoted by γ.) Since p_0 is assumed to be small, we have

$$s \doteq -\beta \log(p_0\gamma'\beta), \quad p_0 \doteq \beta^{-1} e^{-\gamma - s/\beta}. \tag{77}$$

On applying equation (46) we see that

$$\mathscr{E}_N(n_r \mid H_5) \doteq \frac{\beta}{r}\left(\frac{N}{N+\beta}\right)^r \quad (r \geqslant 1, r^2 = o(N)), \tag{78}$$

$$\mathscr{E}_N(n_0 \mid H_5) \doteq \beta E[p_0(N+\beta)] \doteq -\beta \log[\beta_0\gamma'(N+\beta)]. \tag{79}$$

The check may be noticed that equations (77), (78) and (79) are consistent with

$$s = \mathscr{E}(s \mid H_5) = \sum_{r=0}^{\infty} \mathscr{E}_N(n_r \mid H_5).$$

Formula (77) is of some interest, but in most applications both p_0 and s will be largely metaphysical, i.e. observable only within very wide proportional limits.

(vi) The difficulty of determining p_0 would not apply to the same extent if $\alpha = -2$, i.e. for hypothesis H_6. (This hypothesis is fairly appropriate for example (iv) of §8.) We have, by (46),

$$\mathscr{E}_N(n_1 \mid H_6) \doteq \lambda x E[p_0(\beta + N)] \doteq -\lambda x \log \frac{p_0\gamma' N}{x}, \tag{80}$$

$$\mathscr{E}_N(n_r \mid H_6) \doteq \frac{\lambda x^r}{r(r-1)} \quad (r \geqslant 2, r^2 = o(N)), \tag{81}$$

where x and λ, unlike β and p_0, depend on N and are given by

$$x = \frac{N}{N+\beta} \tag{82}$$

and

$$\lambda = \frac{N+\beta}{E(p_0\beta)} \doteq -\frac{N+\beta}{\log(p_0\gamma'\beta)}, \quad \beta p_0 \doteq \exp\left[-\gamma - \frac{N+\beta}{\lambda}\right]. \tag{83}$$

If λ and x can be estimated from a sample, then β and p_0 can be determined by (82) and (83) and s can then be determined from (41), which gives

$$s + \beta = e^{-p_0\beta}/[p_0 E(p_0\beta)] \doteq e^{-p_0\beta}/[-p_0 \log_e(p_0\gamma'\beta)]. \tag{84}$$

In order to estimate λ and x from a sample, one could minimize χ^2, more or less as described above for H_3. For this purpose and for others it may be noted that, by (22),

$$V_N(n_r \mid H_6)/\mathscr{E}_N(n_r \mid H_6) \doteq 1 - \frac{r-1}{2r-1}\frac{1}{2^r}\binom{2r}{r} \quad (r \geqslant 2), \tag{85}$$

$$V_N(n_1 \mid H_6)/\mathscr{E}_N(n_1 \mid H_6) \doteq \frac{\lambda}{N+\beta} \tag{86}$$

$$\doteq \frac{2n_2'}{N} \quad \text{if} \quad x \doteq 1. \tag{87}$$

By comparing (85) with (65) we can get an idea of the smallness of the error arising when calculating χ^2 if (65) is used for hypotheses other than H_3.

Another method of estimating λ and x, rather less efficient, but easier, is the one analogous to that used by Fisher for H_3, namely, we may assume that the expected values of $N-n_1$ and of $S-n_1$ are equal to their observed values, i.e.

$$S-n_1 = \lambda \sum_{r=2}^{\infty} \frac{x^r}{r(r-1)} = \lambda[x+(1-x)\log_e(1-x)] = \lambda(1-e^{-Y}-Ye^{-Y}), \tag{88}$$

$$N-n_1 = \lambda \sum_{r=2}^{\infty} \frac{x^r}{r-1} = -\lambda x \log_e(1-x) = \lambda Y(1-e^{-Y}) = \lambda x Y, \tag{89}$$

$$(S-n_1)/(N-n_1) = Y^{-1} - (e^Y-1)^{-1}, \tag{90}$$

where $x = 1-e^{-Y}$. We may solve (90) iteratively, for Y, i.e. $Y = \lim_{n\to\infty} Y_n$, where $Y_1 = 0$ and, for $n = 1, 2, 3, \ldots$,

$$Y_{n+1}^{-1} = \frac{S-n_1}{N-n_1} + (e^{Y_n}-1)^{-1}. \tag{91}$$

When λ and x are specified, the cumulative sums of $\mathscr{E}_N(n_r \mid H_6)$ can be found by making use of the approximation

$$\sum_{t}^{t\geqslant r} \frac{x^t}{t(t-1)} \doteq xE[-(r-1)\log_e x] - E(-r\log_e x) + \frac{x^r}{2r(r-1)}\left(1 + \tfrac{1}{6}\log_e x - \frac{1}{3r}\right), \tag{92}$$

which will be a very good approximation if the terms involving $\frac{1}{6}\log x$ and $\frac{1}{3r}$ are negligible (cf. equation (72)). If $(1-x)r$ is small while r is large, then we can prove the following approximation:

$$\sum_{t}^{t\geqslant r} \mathscr{E}_N(n_t \mid H_6) \doteq \lambda x\left\{\frac{1}{r} - (1-x)[1-\gamma-\log_e(1-x)-\log_e r]\right\}. \tag{93}$$

If $1-x$ is small but $(1-x)r$ is large, then

$$\sum_{t}^{t\geqslant r} \mathscr{E}_N(n_t \mid H_6) \doteq \frac{\lambda x^r}{(1-x)r^2}. \tag{93A}$$

When in doubt about the accuracy of (93) and (93A) it is best to use (92), the calculation of which is, however, ill-conditioned, so that the error integrals may be needed to several decimal places.

(vii) We now come to the 'less completely formulated' hypotheses. H_7 is discussed by Zipf, especially with $\zeta = 2$ and also in the slightly modified form

$$\mathscr{E}(n_r \mid H) \propto (r^2 - \tfrac{1}{4})^{-1}. \tag{94}$$

(See Zipf (1949, pp. 546–7), where there are further references, including ones to J. B. Estoup, M. Joos, G. Dewey and E. V. Condon.) Yule (1944, p. 55) refers to Zipf (1932) and objects to Zipf's word distributions on two grounds. First Yule asserts that the fits are unsatisfactory, and secondly he points out that (in our notation)

$$N = \mathscr{E}_N(\Sigma r n_r \mid H_7) = \infty \quad \text{if} \quad 1 < \zeta \leqslant 2,$$

while $$c_{2,0} = \int_0^1 p^2 f(p)\,dp = \mathscr{E}(p \mid H_7) = \mathscr{E}\left(\frac{\Sigma r(r-1)n_r}{N(N-1)} \,\middle|\, H_7\right) = \infty \quad \text{if} \quad 2 < \zeta \leqslant 3.$$

(viii) Yule's second objection to H_7 can be overcome by introducing a 'convergence factor', x^r, giving H_8. If H_7 is any good at all for any particular application then x will be fairly close to 1. It would be of interest to specify H_8 in terms of a density function, $f(p)$, by solving the simultaneous integral equations

$$\frac{\lambda x^r}{r^\zeta} = \frac{1}{r!}\int_0^\infty (Np)^r e^{-Np} f(p)\,dp \quad (r = 1, 2, 3, \ldots). \tag{95}$$

If $\zeta = 1$, then H_8 reduces of course to H_3.

(ix) H_9 is of interest mainly because it works so well in examples (ii) and (iii) of §8. Besides its formal similarity to H_8 with $\zeta = 2$, H_9 also resembles H_6, in virtue of equation (81). A disadvantage of not specifying $f(p)$ is that $V_N(n_r \mid H_9)$ cannot be conveniently worked out from (22), though it can always be estimated from (23) with considerably more work. Moreover, a correct specification of $f(p)$ is more fundamental than that of the expected values of the n_r's and is more likely to lead to a better understanding of the structure of the population.

In order to estimate λ and x from a sample, we could use either of the two methods discussed for H_3 and H_6, except that in the method of minimizing χ^2 it would perhaps be best to guess a formula for $V_N(n_r \mid H_9)$, after experimenting with formula (23). We shall not discuss this method further in this section. The second method consists in determining λ and x from the equations

$$N = \lambda \sum_{r=1}^{\infty} \frac{x^r}{r+1} = -\frac{\lambda}{x}[x + \log_e(1-x)], \tag{96}$$

$$S = \lambda \Sigma \frac{x^r}{r(r+1)} = \frac{\lambda}{x}[x + (1-x)\log_e(1-x)], \tag{97}$$

$$\frac{S}{N} = \frac{x + (1-x)\log_e(1-x)}{-x - \log_e(1-x)}. \tag{98}$$

x can be determined either by tabulating the right-hand side of (98) or by writing $x = 1 - e^{-Y}$ and determining Y from the equation

$$Y^{-1} = (1 - e^{-Y})^{-1} - (1 + S/N)^{-1}. \tag{99}$$

Y can be found iteratively by writing $Y = \lim_{n\to\infty} Y_n$, where $Y_1 = 1 + N/S$, and, for $n = 1, 2, 3, \ldots$,

$$Y_{n+1}^{-1} = (1-e^{-Y_n})^{-1} - (1+S/N)^{-1}. \tag{100}$$

Then, by (96), we can find λ from $$\lambda = \frac{xN}{Y-x}. \tag{101}$$

Having determined λ and x we may wish to test how well H_0 agrees with the sample. For this purpose we need to calculate cumulative sums of the expectations of the n_r's. This can be done by means of the approximation

$$\sum_{t}^{t\geqslant r} \frac{x^t}{t(t+1)} \simeq E(-r\log_e x) - \frac{1}{x}E[-(r+1)\log_e x] + \frac{x^r}{2r(r+1)}\left(1+\tfrac{1}{6}\log_e x - \frac{1}{3r}\right), \tag{102}$$

deducible from (92). If $(1-x)r$ is small while r is large, then we have the following approximation:

$$\sum_{t}^{t\geqslant r} \mathscr{E}(n_t \mid H_0) \simeq \lambda x\left\{\frac{1}{r} - (1-x)\left[1-\gamma-\log_e(1-x)-\log_e r\right]\right\}, \tag{103}$$

deducible from and of precisely the same form as (93). An idea of the closeness of this approximation can be obtained from example (ii) below. If $1-x$ is small but $(1-x)r$ is large, then

$$\sum_{t}^{t\geqslant r} \mathscr{E}(n_t \mid H_0) \simeq \frac{\lambda x^r}{(1-x)\,r^2}. \tag{103 A}$$

When in doubt about the accuracy of (103) and (103 A) it is best to use equation (102). (See the remarks following equation (93 A).)

8. *Examples.* In each of the four examples given below we use at least two different methods of smoothing the data. One of these methods is, in each example, the graphical smoothing of $\sqrt{n_r}$ for the smaller values of r and another method is the fitting of one or other of the nine special hypotheses of §7. The discussion of these examples is by no means intended to be complete.

Example (i). *Captures of Macrolepidoptera in a light-trap at Rothamsted.* (Summarized from Williams's data (Corbet *et al.* 1943).) $N = 15{,}609$, $S = 240$.

r	n_r	n_r^{IV}	r	n_r	n_r^{IV}	r	n_r	n_r^{IV} (summed)†
1	35	40	11	2	3·5	21–30	18	15·5
2	11	20·0	12	2	3·2	31–50	16	18·0
3	15	13·2	13	5	3·0	51–70	17	11·4
4	14	9·9	14	2	2·8	71–100	8	11·2
5	10	7·9	15	4	2·6	101–150	9	11·8
6	11	6·6	16	3	2·4	151–200	7	7·4
7	5	5·6	17	3	2·3	201–500	12	16·1
8	6	4·8	18	3	2·1	501–1000	6	4·6
9	4	4·3	19	3	2·0	1001–∞	1	0·9
10	4	3·9	20	4	1·9	2349	1	—

† In future tables this word 'summed' will be taken for granted and omitted.

We now present the results of the calculations; followed by comments. (The columns headed n_r^{IV} in the table above are explained in these comments.)

r	n_r	n_r'	n_r''	n'''	n_r^{iv}	r^*	r^{**}	r^{***}	r^{****}
1	35	35	35	35	40	1·1	1·4	1·3	1
2	11	19·4	24·0	22·5	20·0	2·1	2·3	2·2	2
3	15	13·7	18·1	16·3	13·3	3·0	2·9	3·0	3
4	14	10·2	13·1	12·3	10·0	3·8	3·8	3·9	4
5	10	7·8	10·2	9·7	7·9	4·8	4·8	4·8	5
6	11	6·3	8·1	7·7	6·6	5·9	5·9	5·5	6
7	5	5·3	6·8	6·0	5·6	—	—	—	—

The function n_r' was obtained by plotting $\sqrt{n_r}$ against r for $1 \leqslant r \leqslant 20$ and smoothing for $1 \leqslant r \leqslant 7$ by eye, holding in mind the method of least squares. (See note (i) of §3.) n_r'' was obtained in the same way, but an attempt was made to keep away from the graph of n_r' (except at $r = 1$) in order to find out how different a smoothing was reasonable. Next n_r''' was obtained by smoothing the cumulative sums $\sum_{i=1}^{r} tn_t$. Finally, n_r^{iv} is the function obtained by Fisher, i.e. using our hypothesis H_3 (equation (63)) with $\beta = 40{\cdot}2$ and $x = 0{\cdot}9974$. A more complete tabulation of n_r^{iv} is given in the first table. The 'summed' values of n_r^{iv} were calculated by means of equation (72). No statistical test is necessary to see that the fit of n_r^{iv} is very good. The values of r^* corresponding to the four smoothings of the data are denoted by r^*, r^{**}, r^{***} and r^{****} respectively. (Logically this gives r^* two different meanings.) ($r^{****} = 0{\cdot}9974r$, by (2′) and (63).) In accordance with §3 we could force the r^*'s, etc., to be smooth. This has not been tried here. What is clear is that if H_3 is not accepted then most of the values of r^*, etc., are unreliable to within about 0·2 or 0·3. The approximate values of χ^2 given by (19) with $r = 7$ and assuming (65) are 10·9, 11·1, 9·4 and 11·7 respectively. The number of degrees of freedom is somewhere between 6 and 7. It seems safe to take it as 5 for n_r''', 6 for n_r' and n_r'' and 7 for n_r^{iv}. None of the values of χ^2 is particularly significant, though all are a bit large. The data can be blamed for the largeness of the values of χ^2, since n_2 is obviously much smaller than it ought to be. Of the four smoothings Fisher's seems to be the most likely to give the best approximations to the 'true expectations'. There is hardly anything to choose on the evidence of the sample, but Fisher's smoothing has the advantage of being analytically simple.

The most definite result of interest in this example does not depend much on the smoothing, namely, that the proportion of the population not represented by the species in the sample is about $(35 \pm 5)/15{,}609$. For the '± 5' see formula (65). Perhaps this standard error should be increased slightly, say from 5 to 8, to allow for the preference given to n_r^{iv}.

Formula (77), if it is applicable (i.e. if the truncated form, H_5, of H_3 is assumed), may be written $-\log_{10} p_0 = 1{\cdot}18 + 0{\cdot}011s$, so that if s were say 1000, then the smallest population frequency would be about 10^{-12}. This is mentioned only for its theoretical interest: it is an unjustifiable extrapolation to suppose that the distribution defined by H_5 would stand up to sample sizes large enough to demonstrate clearly the values of s and p_0. N would need to be of the order of $10/p_0$. The proposition which is made probable by the actual sample is that H_3 and H_5 (with the assigned values of the parameters) would give good fits to the values of n_r on other independent samples of 16,000 or less, i.e. that H_3 and H_5 provide good methods of smoothing the data. The cautious tone of this statement can be more fully justified by the following considerations.

If H_5 were reliable then it should be possible to use it to estimate the simpler measures of heterogeneity, such as $c_{2,0}$. Now we can see by (30) that $\hat{c}_{2,0} = 0{\cdot}03935$ and $\hat{c}_{3,0} \doteq 0{\cdot}0035$. (For the calculations, the complete data given by Williams must be used.) Hence, by (30A), it is reasonable to write $c_{2,0} = 0{\cdot}03935 \pm 0{\cdot}0007$. Let us then see what value for $c_{2,0}$ is implied by H_5. We have

$$\Sigma r(r-1)\, n_r^{\text{iv}} = \beta\Sigma(r-1)\, x^r = \beta x^2/(1-x)^2 = 0{\cdot}0243.$$

[As a check, $\displaystyle\int_0^\infty p^2 f(p)\, dp = \beta \int_{p_*}^\infty p\, e^{-\beta p}\, dp = \beta \int_{p_*\beta}^\infty q\, e^{-q} dq \doteq \beta^{-1} = 0{\cdot}025.$]

Clearly then H_5 cannot be used to estimate $c_{2,0}$. It would be true if misleading to say that H_5 is decisively disproved by the data. Similar remarks would apply in the examples below.

Example (ii). *Eldridge's statistics for fully inflected words in American newspaper English.* Eldridge's statistics (1911) are summarized by Zipf (1949, pp. 64 and 25). We give a summary of Zipf's summary in column (ii) below; more fully in the second table. $N = 43{,}989, S = 6{,}001$.

In this example the values of n_r for $r \leqslant 10$ are much larger than in example (i), so we have far more confidence in the smoothing that is independent of particular hypotheses. We shall present some of the numerical calculations in columns and then make comments on each column. We may assert at once, however, by equations (7), (8) and (9), that the proportion of the population represented by the sample is close to $1 - n_1/N = 14/15$. If a foreigner were to learn all 6001 words which occurred in the sample he would afterwards meet a new word at about 6·7 % of words read. If he learnt only $S - n_1 = 3025$ words he would meet a new word about 11·6 % of the time. The corresponding results for word-roots rather than for fully inflected words would be of more interest to a linguist.

(i) r	(ii) n_r	(iii) $\sqrt{n_r}$	(iv) b_r say	(v) $-\Delta$	(vi) rb_r	(vii) rb'_r say	(viii) b'_r	(ix) b'^2_r	(x) n'_r	(xi) r^*
1	2976	54·5	54·5	21·8	54·5	54·5	54·5	2976	2961	0·73
2	1079	32·7	32·7	10·0	65·4	65·4	32·7	1079	1075	1·4
3	516	22·7	22·7	5·7	68·1	67·8	22·6	511	509	2·4
4	294	17·1	17·0	2·8	68·0	70·2	17·5	206	305	3·4
5	212	14·6	14·2	1·8	71·0	72·6	14·5	210	209	4·4
6	151	12·3	12·4	1·5	74·4	74·7	12·4	154	153	5·4
7	105	10·2	10·9	1·3	76·3	76·3	10·9	119	118	6·2
8	84	9·2	9·6	1·2	76·8	76·8	9·6	92	91	—
9	86	9·3	8·4	1·1	75·6	75·6	8·4	71	70	—
10	45	6·7	7·3	—	—	—	—	—	—	—

(i) and (ii). We first consider the values of r only as far as $r = 10$. For larger values of r the smoothing could be done by using k-point smoothing formulae with $k \doteq 2\sqrt{r}$.

(iii) Each entry in this column has standard error of about $\frac{1}{2}$, so one place of decimals is appropriate.

(iv) This column was obtained by smoothing a graph of column (iii) by eye. Experiments with the five-point smoothing formula did not give quite as convincing results. For the five-point smoothing formula, see, for example, Whittaker & Robinson (1944, §146). For the present application it would be $\sqrt{n'_r} = \sqrt{n_r} - \frac{3}{35}\Delta^4(\sqrt{n_r})$ $(r = 3, 4, 5, \ldots)$.

(v) This column of differences is given as a verification of the smoothness of column (iv). In fact minor adjustments were made in column (iv) in order to improve the smoothness of column (v).

(vi) The numbers b_r of column (iv) are roughly proportional to r^{-1}. This fact suggests that rb_r should be formed and smoothed again in order to improve the smoothing of $\sqrt{n_r}$ still further. This process is of course distinct from assuming that rb'_r should be constant, where the function b'_r is a smoothing of the function b_r.

(vii) and (viii) These columns have already been partly explained. The purpose of this improvement in the smoothing is more for the sake of the ratios n'_{r+1}/n'_r than of the n'_r themselves.

(ix) Where the smoothing of $\sqrt{n_r}$ had no noticeable effect we have taken $b'^2_r = n_r$. It is clearly typical that $b'^2_1 = n_1$, since the eye-smoothing is unlikely to affect n_1 convincingly. Therefore if the smoothing is tested by means of a chi-squared test it will be reasonable to subtract about two degrees of freedom.

(x) We have scaled up column (ix) so as to force $\sum_{r=1}^{9} rn'_r = \sum_{r=1}^{9} rn_r$. We can then assume $N' = N$, convenient for applications of §6. Note that $\sum_{r=1}^{9} k_r(n'_r - n_r)^2/n'_r = 6{\cdot}5$, so that χ^2, given by (19) and accepting (65) as a good enough approximation, is not significant on eight degrees of freedom. Thus our smoothing is satisfactory, though there may be other satisfactory smoothings.

(xi) r^* is obtained from formula (2′). The larger is r the larger is the standard error of r^*. We may get some idea of the error by means of an alternative smoothing. The standard error of 1^* can be very roughly calculated by an *ad hoc* argument, inapplicable to say 5^*. We may reasonably say that the variance of $2n'_2/n'_1$ with respect to all eye-smoothings will be about the same as that obtained by regarding n'_2 and n'_1 as independent random variables with variances circumscribed by the inequalities (26) and (27), or nearly enough, defined by (65). Now if w and z are independent random variables with expectations W and Z, we have

$$\delta\left(\frac{w}{z}\right) = \frac{\delta w}{Z} - \frac{W\,\delta z}{Z^2},$$

and hence, to a crude approximation,

$$V\left(\frac{w}{z}\right) \simeq \frac{V(w)}{Z^2} + \frac{W^2 V(z)}{Z^4},$$

i.e.
$$\frac{V(w/z)}{W^2/Z^2} \simeq \frac{V(w)}{W^2} + \frac{V(z)}{Z^2}. \tag{104}$$

It follows that
$$\frac{V(1^*)}{1^{*2}} = \frac{V(2n'_2/n'_1)}{(2n'_2/n'_1)^2} \simeq \frac{1}{k_1 n'_1} + \frac{1}{k_2 n'_2}, \tag{105}$$

so that $V(1^*) = 0{\cdot}73^2 \times 0{\cdot}0010 = 0{\cdot}00052$ and $1^* = 0{\cdot}73 \pm 0{\cdot}023$.

(xii) (see the second table). An analytic smoothing which is remarkably good for $r \leqslant 15$ is given by $n''_r = S/(r^2 + r)$. For larger values of r there is a serious discrepancy, since $\sum_{r=16}^{\infty} n''_r = 374$ while $\sum_{r=16}^{\infty} n_r = 297$. It is clear without reference to the sample that n''_r cannot be satisfactory for sufficiently large values of r, since $\Sigma r n''_r = \infty$ instead of being equal to N.

(i) r	(ii) n_r	(x) n'_r	(xii) n''_r	(xiii) n'''_r	(xi) r^*	(xiv) r^{**}
1	2976	2961	3000	3008	0·73	0·67
2	1079	1075	1000	1002	1·4	1·5
3	516	509	500	500	2·4	2·4
4	294	305	300	301	3·4	3·3
5	212	209	200	201	4·4	4·3
6	151	153	143	144	5·4	5·2
7	105	118	107	108	6·2	6·2
8	84	91	83	84	—	7·2
9	86	70	67	67	—	8·2
10	45	—	55	55	—	—
11–15	156	—	170	170	—	—
16–20	76	—	89	89	—	—
21–30	78	—	92	92	—	—
31–40	34	—	47	47	—	—
41–50	28	—	29	28	—	—
51–60	10	—	19	19	—	—
61–∞	71	—	98	90	—	—
4290	1	—	—	—	—	—

(xiii) The fit can be improved by writing $n'''_r = \lambda x^r/(r^2+r)$ as in equation (55), i.e. using hypothesis H_9. We find by equations (100) and (101) that $\lambda = 6017{\cdot}4$ and $x = 0{\cdot}999667$. Column (xiii) can then be easily calculated directly for $r \leqslant 10$ and by use of (102) or (103) for $r > 10$. ((103) gives the correct values for n'''_9 and n'''_{10}, to the nearest integer, and it gives $\sum_{61}^{\infty} n'''_r = 89{\cdot}96$, as compared with 89·90 when (102) is used.) Note that $\sum_{r=16}^{\infty} n'''_r = 365$, which implies an improvement on n''_r but is still significantly too large. A better fit could be obtained by the method of minimum χ^2 or by using some simple convergence factor other than x^r, such as e^{-ar-br^2} with $a > 0$, $b > 0$.

(xiv) r^{**} is defined as $(r+1)\,n_{r+1}/n_r$ and is equal to $r(r+1)/(r+2)$. This column may be compared with column (xi). The agreement looks fairly good. It is by no means clear which of the two columns gives more reliable estimates of the 'true' values of r^* for $r \leqslant 7$. Column (x) is a better fit to Eldridge's data for $r \leqslant 9$ (and could be extended to be a better fit for all r) than is column (xiii) but is not as smooth. Columns (xii) and (xiii) would be preferable if some theoretical explanation of the analytic forms could be provided. Such an explanation might also show why the fit is not good for large r, even with the convergence factor x. The limitation on r in equation (46) may be relevant.

If H_9 is true, the population parameter $c_{2,0}$, given by (31), can be expressed in the form

$$\frac{\lambda}{N^{(2)}} \sum_{r=1}^{\infty} \frac{r-1}{r+1} x^r = \frac{\lambda}{N^{(2)}} \left[\frac{x}{1-x} - 2 + \frac{2}{x} \log_e (1-x) \right]. \tag{106}$$

Formula (106) would give $c_{2,0} = 0{\cdot}00928$, but this value is probably a bad over-estimate since n'''_r is too large for large r and the terms of $\frac{\lambda}{N^{(2)}} \Sigma \frac{r-1}{r+1} x^r$ for large r make most of the contribution. Similarly, $\hat{c}_{2,0}$, given by (30), depends mainly on the larger values of r represented in the sample, but Zipf's summary of Eldridge's data is not complete enough to calculate $\hat{c}_{2,0}$. Similarly, assuming H_9, the entropy, $c_{1,1}$, could be estimated from equation

(39), and this method could be expected to give close agreement with the correct value, since $c_{1,1}$ does not depend so very much on the more frequent species. But I have not obtained a closed formula, resembling (106) for example, and the arithmetic required if no closed formula is available would be heavy. The estimation of measures of heterogeneity will be discussed again under example (iii).

Example (iii). *Sample of nouns in Macaulay's essay on Bacon.* (Taken from Yule (1944) Table 4·4, p. 63.) $N = 8045$, $S = 2048$.

r	n_r	r	n_r	r	n_r	r	n_r	r	n_r
1	990	11	24	21	1	31	2	41	1
2	367	12	19	22	4	32	1	45	2
3	173	13	10	23	7	33	1	48	1
4	112	14	10	24	2	34	1	57	1
5	72	15	13	25	1	35	1	58	1
6	47	16	3	26	5	36	1	65	1
7	41	17	10	27	3	37	1	76	1
8	31	18	7	28	4	38	2	81	1
9	34	19	6	29	1	39	4	89	1
10	17	20	5	30	3	40	1	255	1

As in example (ii) we can state some conclusions at once, without doing the smoothing. If our foreigner learns all 2048 nouns that occur in the sample his vocabulary will represent all but $(12{\cdot}3 \pm 0{\cdot}5)$ % of the population, assuming formulae (9) and (65) or (87). If he learns only 1058 nouns his vocabulary will still represent all but $(n_1 + 2n_2)/N = 19{\cdot}3$ % of the population.

We now present three different smoothings corresponding precisely to those of example (ii).

r	n_r	n'_r	n''_r	n'''_r	r^*	r^{**}	r^{***}	$\frac{d}{dr}\log_{10} n'_r$	$\frac{d}{dr}\log_{10} n'''_r$	$g_r \log_{10} e$
1	990	990	1024	1060	0·74	0·67	0·66	−0·50	−0·65	0·184
2	367	367	341	350	1·4	1·5	1·5	−0·30	−0·37	0·401
3	173	173	170	174	2·6	2·4	2·4	−0·24	−0·26	0·545
4	112	112	102	103	3·4	3·3	3·3	−0·17	−0·20	0·654
5	72	76	68	68	4·4	4·3	4·3	−0·15	−0·16	0·741
6	47	56	49	48	5·3	5·2	5·1	−0·12	−0·14	0·813
7	41	42	35·5	36	6·5	6·2	6·1	−0·11	−0·12	0·876
8	31	34	28·5	28	7·3	7·2	7·1	−0·10	−0·11	0·930
9	34	27	22·7	22	8·2	8·2	8·1	−0·09	−0·10	0·978
10	17	22	18·4	18	—	9·2	9·1	−0·08	−0·09	1·021
11	24	18·5	15·5	15	—	10·2	10·1	—	—	—
12	19	16·0	13·1	12	—	11·1	11·0	—	—	—
13	10	13·7	11·3	10	—	12·1	12·0	—	—	—
14	10	10·9	9·7	9	—	13·1	13·0	—	—	—
15	13	9·6	8·5	8	—	14·1	14·0	—	—	—
16–20	31	32·5	30·5	27	—	—	—	—	—	—
21–30	31	—	31·5	26	—	—	—	—	—	—
31–50	19	—	25·9	19	—	—	—	—	—	—
51–100	6	—	19·9	11	—	—	—	—	—	—
101–∞	1	—	20·3	3·6	—	—	—	—	—	—
255	1	—	—	—	—	254	252	—	—	—

$\sqrt{n'_r}$ was obtained by smoothing $\sqrt{n_r}$ graphically.

$n_r'' = S/(r^2+r)$. It is curious that this should again give such a good fit for values of r that are not too large ($r \leqslant 30$). The sample is of nouns only and, moreover, Yule took different inflexions of the same word as the same.

$n_r''' = \lambda x^r/(r^2+r)$, where $\lambda = 2138{\cdot}90$, $x = 0{\cdot}991074$, the values being obtained from (100) and (101) as in example (ii).

The expressions $\sum_{r=1}^{15}(n_r'-n_r)^2/n_r'$, etc., take the values 9·5, 21·2 and 27·3. The values of χ^2 would be about 2 or 3 larger. (See (19), (26), (27), (65).) There is no question of accepting n_r'' for $r > 50$ but it is better than n_r''' for $r \leqslant 15$. When $r \leqslant 9$ the values of r^* and r^{**} (and therefore of r^{***}) show good agreement except for $r = 1$ and $r = 7$. If the analytic smoothings had not been found, the value of 6* would have been smoothed off, with repercussions on the function n_r'. The discrepancy in 1* must be attributed either to a fault in the value of n_1''' (and therefore in H_9) or must be blamed on n_1 (i.e. on sample variation). If I had not noticed the analytic smoothings I would have asserted that 1* = 0·74 with a standard error of something like 0·04. (See equation (105).)

We now consider two of the measures of heterogeneity in the population, namely, $c_{2,0}$ and $c_{1,1}$. By (30) we can see that $\hat{c}_{2,0} = 0{\cdot}00272$, agreeing with Yule (1944, p. 57). Also $\hat{c}_{3,0} = 0{\cdot}00003957$, so that by (30A) we may reasonably write $c_{2,0} = 0{\cdot}00272 \pm 0{\cdot}00013$. Assuming H_9 to be valid for $r \leqslant 30$, we may also estimate $c_{2,0}$ by $\tilde{c}_{2,0}$ (30) as in equation (33). We have, in a self-explanatory notation,

$$\tilde{c}_{2,0}(30 \mid H_9) = \frac{1}{N^{(2)}}\left(\lambda \sum_{1}^{30}\frac{r-1}{r+1}x^r + \sum_{30}^{\infty} r^{(2)}n_r\right). \tag{107}$$

Now, as in (72),

$$\sum_{31}^{\infty}\frac{r-1}{r+1}x^r \doteq \frac{x^{31}}{1-x} - \frac{2}{x}\left\{E(-32\log_e x) + \frac{x^{32}}{64}\left(1+\tfrac{1}{6}\log_e x - \tfrac{1}{192}\right)\right\} = 82{\cdot}924.$$

But, as in (106), $\sum_{1}^{\infty}\frac{r-1}{r+1}x^r = 99{\cdot}501$, so that $\sum_{1}^{30}\frac{r-1}{r+1}x^r = 16{\cdot}577$. It follows from (107) that $\tilde{c}_{2,0}(30 \mid H_9) = 0{\cdot}00246$. This is about two standard errors below its expected value, based on the simple unbiased statistic $\hat{c}_{2,0}$. The discrepancy may again be attributed to the large value of n_1'''. If, instead of n_r''', the smoothing n_r'' is accepted for $r \leqslant 30$, we would get $\tilde{c}_{2,0}(30) = 0{\cdot}00267$. (It was in order to obtain this comparison that we calculated $\tilde{c}_{2,0}(30 \mid H_9)$ rather than $\tilde{c}_{2,0}(50 \mid H_9)$. The fit of n_r'' deteriorates at about $r = 30$.)

The last three columns of the table are related to the estimation of the entropy, $c_{1,1}$. (See equation (40) and the remarks following it.) $\frac{d}{dr}\log_{10} n_r'$ was obtained graphically for $r = 1$, 2 and 3 by numerical differentiation for $r = 3, 4, \ldots, 10$. (The graphical and numerical values agreed to two decimal places for $r = 3$.) The column $\frac{d}{dr}\log_{10} n_r'''$ was of course calculated as $\log_{10} x - \left(\frac{1}{r}+\frac{1}{r+1}\right)\log_{10} e$. The crude estimate of the 'entropy to base 10' or 'entropy expressed in decimal digits' is $\log_{10} N - \frac{1}{N}\sum_r r n_r \log_{10} r = 2{\cdot}968$ decimal digits. If n_r' is accepted for $r = 1, 2, 3, \ldots, 10$ we find that

$$\tilde{c}_{1,1}(10) = \log_{10} N - \frac{1}{N}\left\{\sum_{r=1}^{10} r n_r'\left(g_r \log_{10} e + \frac{d}{dr}\log_{10} n_r'\right) + \sum_{r=11}^{\infty} r n_r \log_{10} r\right\} = 3{\cdot}051 \text{ decimal digits.}$$

We shall next calculate $\tilde{c}_{1,1}(50 \mid H_9)$, using another self-explanatory notation. Since, by Jeffreys & Jeffreys (1946, §15·05),

$$g_r \sim \log_e r + \frac{1}{2r} - \frac{1}{12r^2} - \ldots,$$

it can be seen that

$$\tilde{c}_{1,1}(50 \mid H_9) \doteq \log_{10} N - \frac{1}{N}\left\{\sum_{r=1}^{10} r n_r''' \left(g_r \log_{10} e + \frac{d}{dr}\log_{10} n_r'''\right) + \sum_{11}^{50} r n_r''' \log_{10} r + \log_{10} x \sum_{11}^{50} r n_r''' - \frac{3\log_{10} e}{2}\sum_{11}^{50} n_r''' + \sum_{51}^{\infty} r n_r \log_{10} r\right\}$$

$$= 3{\cdot}192 \text{ decimal digits},$$

as we may see by means of rather heavy calculations, using the last column of the table, together with equations (72), (74) and (92). The crude estimate of $c_{1,1}$ is the smallest of the three. This is not surprising, since the crude estimate is always too small in the special case of sampling from a population of s species all of which are equally probable.

Example (iv). *Chess openings in games published in the British Chess Magazine*, 1951. For the purposes of this example we arbitrarily regard the openings of two games as equivalent only if the first six moves (three white and three black) are the same and in the same order in both games. $N = 385$, $S = 174$.

r	n_r	n_r'	n_r''	n_r'''	r^{**}
1	126	126	126	126	0·39
2	22	22	24·6	24	1·0
3	5	7·6	8·1	8	2·0
4	4	4·8	4·0	4	3·0
5	3	3·2	2·4	2·4	4·0
6	4	2·6	1·6	1·6	5·0
7	0	2·2	1·1	1·14	6·0
8	3	—	0·85	0·86	7·0
9	1	—	0·66	0·67	8·0
10	1	—	0·52	0·53	9·0
11	0				
13	1				
14	1				
16	1		3·97	4·80	
23	1				
36	1				
∞	—				

$\sqrt{n_r'}$ was obtained by graphical smoothing of $\sqrt{n_r}$.

n_r'' was obtained by assuming H_6 (see equation (52)), i.e. $n_r'' = \mathscr{E}_{385}(n_r \mid H_6)$, where the parameters x and λ were obtained from (91) and (89). These gave $x = 0{\cdot}99473$, $\lambda = 49{\cdot}635$ and n_r'' for $r \geqslant 2$ is then given by (81). Next p_0 was determined as $0{\cdot}00011304 = 1/8846$ by using equation (80). Then (82) gave $\beta = 2{\cdot}040$, so that, in accordance with (52) and (74),

$$f(p) \doteq \begin{cases} 0{\cdot}128 p^2 e^{-2{\cdot}040p} & (p > 1/8846), \\ 0 & (p < 1/8846). \end{cases}$$

Finally, equation (84) gives $s = 1132$. This then is the estimate of the total number of openings in the population, though the sample is too small to put any reliance in it.

$$n_r'''(r \geqslant 2) \text{ is simply } (S-n_1)/(r^2-r) = 48/(r^2-r).$$

This is just as good a fit as n_r''. It gives an infinite value to $c_{2,0}$, but this is not as serious an objection as it sounds since H_6 would also give quite the wrong value for $c_{2,0}$. (Cf. the concluding remarks in the discussion of example (i).)

We list in the table the values of r^* corresponding to n_r'', calling the values r^{**} in conformity with the convention of the present section. Clearly $r^{**} = (r-1)x$ when $r \geqslant 2$. Thus the average population frequency of the 126 openings that each occurred once only in the sample is $0{\cdot}39/385 = 0{\cdot}001$.

A player who learnt all 174 openings would expect to recognize about 67 % of future openings for the same population, assuming that the sample was random. If he learnt the 48 openings that each occurred twice or more in the sample the percentage would drop to 55 % and if he learnt the 26 that occurred three times or more the percentage would drop to 49 %. (See formula (6′).)

9. *Index of notations having a fixed meaning.*

§1. N, n_r (but see also §2), n_0, q_r, r^* (as a definition of the asterisk, but there is a slight change of convention in §8), n_r' (here again there is a slight change in §8), $\mathscr{E}(\ ), V(\ )$.

§2. $s, p_\mu, H(p_1, p_2, \ldots, p_s) = H, \mathscr{E}_N, \mu'_{r,t,N}$.

§3. N'.

§5. $x_{\mu,r} = x_\mu, a_\mu$.

§6. $c_{m,n}, \hat{c}_{m,0}, \tilde{c}_{m,0}, \tilde{c}_{m,0}(t), \gamma, g_r, \hat{\tilde{c}}_{1,1}, \tilde{c}_{m,1}(t)$.

§7. $p, f(p), p_0, H_1$ to $H_9, E(\ ), k_r, S, \gamma'$.

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CAPTURE-RECAPTURE ANALYSIS

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This paper presents a new method for the analysis of capture-recapture data. Previous methods have employed deterministic or partially stochastic models of the population under study; the new method not only provides a fully stochastic model, but it also allows death-rates to depend upon both current time and the health and age of the separate individuals. The numerical analysis associated with the method is, however, heavy, and can only be recommended when one wants to extract the maximum amount of information from data which have required much effort or time to collect. The method is illustrated by an estimation of the death-rate of Alpine Swifts (*Apus melba*) in the wild state. Some thirty years having been spent in collecting these data, it does not seem disproportionate to have spent several months in computing the results. The larger part of this calculation was done on ordinary desk calculators.

Some unusual problems in maximum-likelihood estimation are discussed.

Introduction

The literature upon capture-recapture analysis is now extensive.* To acquaint himself with it, the reader may consult the references given at the end of this paper, and then follow up the references cited in turn in these papers. To make the present paper fairly self-contained, however, we begin with a brief outline of the problem.

The main object is to estimate the (possibly changing) size of a population of individuals (e.g. animals, birds, fish, insects, etc.) in their natural state. To this end, the experimenter captures individuals from the population on a number of successive occasions. On each occasion the catch is to be considered as a random sample of individuals from the population; that is to say, each individual in the population (if alive) has an equal chance of being captured on any given occasion irrespective of its age, health, type, etc., and of any previous captures it may have suffered, although this chance may vary from one occasion to another. Each time an individual is caught, a record is marked for it or on it to show the occasion of that capture; the individual is then returned to the population. Each individual in the population may die at any time, the chance of death being supposed to depend only upon the current time, and the current age, type and condition of the individual. The experimenter has to estimate the size of the population at any given instant from records of the several occasions of capture. The main difficulty is that this estimation depends upon an estimate of the effective death-rate then prevailing in the population, and that this latter estimate is hard to make.

The experimenter may or may not be able to distinguish between one individual and another. For instance, in the case of birds, he can place a numbered ring upon the leg of each captured bird; and, if all rings bear distinct numbers, he will be able to identify any particular bird upon every occasion that it is subsequently recaptured. On the other hand, in the case of moths, the marks may be only spots of paint on the wing, the colour and position of the marks indicating the dates of capture but failing to discriminate between two moths which have both suffered the same history of captures. Merely for convenience in describing the theory, we shall suppose that the experimenter has fastened individually

* Laplace had ideas on the subject 170 years ago, and the last 60 years have produced a steady flow of papers.

numbered rings upon all individuals at their first capture; but we shall later see that as a rule the analysis can still be carried through in the other case.

Let us then suppose that the experimenter captures individuals at a sequence of discrete instants $T_1, T_2, \ldots, T_m$; and that the individuals which are captured at some time or another are named $I_1, I_2, \ldots, I_n$ by means of their distinct rings numbered $1, 2, \ldots, n$. The complete set of data then takes the form of an $m \times n$ matrix in which the element in row i and column j is 1 or 0 according as the individual I_j was or was not captured (or recaptured) at the instant T_i. Call this data matrix $\mathbf{D} = (d_{ij})_{mn}$.

There are two ways of analysing **D**. According to the first scheme, which we call 'analysis by rows', we first consider any given subset of rows of **D**, and count the number of columns which contain 1 in every row of the subset and zero in every row not belonging to the subset. Let us call this the α-number of the subset. We then analyse the set of all α-numbers obtained by considering every possible subset of rows. This method leads to very considerable algebraic difficulties since the α-numbers so obtained are not independent of each other, and therefore cannot be combined in a simple fashion to yield the likelihood of **D**. Analysis by rows is basically the structure which underlies most previous attacks on the capture-recapture problem; but, either because of the algebraic difficulties just mentioned or because of limitations in the data presented to them, previous authors have adopted one or more of the following simplifying expedients:

(*a*) Certain subsets of rows are grouped together according to some rule of procedure, and the analysis is confined to the totals of the α-numbers falling in the groups thus generated. This pooling of α-numbers may perhaps sacrifice some information. Also it will be dictated to some extent should the available data not give complete records of individual histories.

(*b*) No use is made of the α-numbers corresponding to subsets which comprise a single row only; that is to say, we disregard those individuals which are captured only once and never recaptured. This sacrifices information because these special α-numbers influence the estimates of the experimenter's catching efficiency, on the precision of which estimates depends also the precision of the death-rate estimate. Again this will be dictated if the data does not contain records of individuals that are never recaptured.

(*c*) It is assumed that the total catch at any given instant of catching is a small fraction of the population at risk of capture; and that therefore the distribution of the α-numbers, which is strictly a multihypergeometric distribution, may be replaced by a multinomial distribution.

(*d*) It is assumed that death-rates operate deterministically; that is to say, the proportion of birds, having any given history of captures, which will die in a specified period of time is not subject to random variation.

(*e*) It is assumed that death-rate is independent of age and current time.

The alternative scheme is 'analysis by columns'. This simple change of viewpoint (which, for the analysis of the Alpine Swifts, is in a sense to take a bird's-eye view of the data!) sweeps away all the algebraic difficulties and allows us to avoid, if we so desire and if the comprehensiveness of the data permits, any or all of the palliatives (*a*)–(*e*) above. Although analysis by columns has these algebraic advantages, it has, on the other hand, the disadvantage that it leads (as might be expected) to more complicated equations and considerably heavier computing than in the case when analysis by rows is undertaken with one or more of the assumptions (*a*)–(*e*). Analysis by columns involves finding separately

for each column of **D** the likelihood of the observations in that column. Then the likelihood of **D** is the product of these separate likelihoods; for the histories of two individuals are mutually independent under the assumptions made in the second paragraph of this introduction. The most serious restriction in these assumptions is probably that the chance of death depends only upon current time and current age, type and condition of the individual; for in reality the death of one individual may affect the chances of another's survival, as in case of over-population.

One other artifice, which will be fundamental, is the

THEOREM. *Parameter spaces may be transformed before or after maximum-likelihood estimation with the same final result in either case.*

This is certainly not an original statement, and it is quite obvious once made. It is, of course, false for most forms of estimation other than maximum likelihood. Statisticians, to whom I have mentioned it, fall into two classes of roughly equal size: those of one class knew it from their cradles, those of the other show momentary disbelief. I would have belonged to the second class had not this capture-recapture analysis forced the theorem to my notice.

The analysis has thrown up a further aspect of maximum-likelihood estimation. Suppose that the likelihood L depends upon a parameter $\boldsymbol{\theta}$ with co-ordinates $\theta_1, \theta_2, \ldots, \theta_q$. Then as a rule the maximum-likelihood solution is obtained as the solution of the simultaneous equations

$$\partial L/\partial\theta_j = 0 \quad (j = 1, 2, \ldots, q). \tag{1}$$

What should one do, however, if no solution exists for these equations, or (worse) if the equations themselves do not exist (for example, the likelihood function may not be differentiable with respect to θ_j)? Even the rigorous text-books seem to overlook this possibility; and yet it may arise in the very simplest of practical problems. For instance, suppose that we wish to estimate the parameter π of a binomial distribution having performed N trials of which n are 'successful'. The likelihood function is

$$L = \binom{N}{n} \pi^n (1-\pi)^{N-n},$$

and hence

$$\frac{1}{L}\frac{\partial L}{\partial \pi} = \begin{cases} -N/(1-\pi) & (n=0), \\ (n-N\pi)/\pi(1-\pi) & (0<n<N), \\ N/\pi & (n=N). \end{cases}$$

When $0<n<N$, we get the usual solution $\pi = n/N$; but when $n = 0$ or N, the equation $\partial L/\partial\pi = 0$ has no solution. In these cases it seems reasonable to take $\pi = 0$ and $\pi = 1$ respectively; for, subject to the known information $0 \leqslant \pi \leqslant 1$, these values of π make the likelihood as large as possible. It is easy, however, to construct examples in which no value of the parameter makes the likelihood as large as possible. For instance, consider the foregoing binomial distribution, when $n = 0$ and we have the prior restriction $0 < \pi$. In general the following procedure may be acceptable to some readers, and I, for one, would accept it for want of a better. Let $\boldsymbol{\theta}$ be an unknown vector parameter restricted to a known set Θ, and let $\overline{\Theta}$ denote the closure of Θ. Since the likelihood function satisfies $0 \leqslant L \leqslant 1$, it possesses a supremum $\overline{L}$ for fixed data and $\boldsymbol{\theta}$ belonging to Θ. Let $\epsilon > 0$ be prescribed and let $\overline{\Theta}(\epsilon)$ denote the closure of the set of all $\boldsymbol{\theta}$ in Θ such that $L \geqslant \overline{L} - \epsilon$ for the fixed data. Then

$\overline{\Theta}(\epsilon_1) \supseteq \overline{\Theta}(\epsilon_2)$ whenever $\epsilon_1 \geqslant \epsilon_2$; and hence $\overline{\Theta}(0) = \lim_{\epsilon \to 0+} \overline{\Theta}(\epsilon)$ exists in $\overline{\Theta}$. It is to be noticed that $\overline{\Theta}(0)$ may not have any points in Θ, it may not be a unique point or even a countably infinite set of points, and that the likelihood for any $\boldsymbol{\theta}$ in $\overline{\Theta}(0)$ may be less than every L for $\boldsymbol{\theta}$ in Θ. It is, indeed, possible to construct an example in which the likelihood is not constant and yet, for almost all data, every point of the parameter space is simultaneously a maximum-likelihood and a minimum-likelihood solution; for example, it suffices to take L equal to a positive constant or zero according as θ is rational or not, except that L is to be suitably modified for one particular value of the data (thus preventing a solution which is independent of the data). Nevertheless, I should be prepared to define $\overline{\Theta}(0)$ as the maximum-likelihood solution, mainly on the grounds that it reduces to the conventional maximum-likelihood solution whenever this latter exists, and that in many other cases it seems to produce the sensible answer. The problem now remains of specifying some expression for the variance of this estimate; but I have no satisfactory idea how to set about this. In the first place one has to define what may be meant by the variance of an uncountable statistic. Secondly, even when the statistic is unique, one has to determine what should replace the conventional expression $\partial^2 \log L / \partial\theta_i \, \partial\theta_j$. Presumably if one can calculate a variance-covariance matrix $\mathbf{V}$ for any unique $\boldsymbol{\theta}$, then what is wanted in general is an expression of the type $M\mathbf{V}$, where M is a generalized averaging over $\overline{\Theta}(0)$. Thus, if $\overline{\Theta}(0)$ is measurable and not of zero measure, M might denote normalized integration with respect to this measure. Alternatively, one might prefer an averaging defined as the extraction of a supremum or infimum; and a number of other possibilities will suggest themselves to the reader. The only precaution to observe is that the averaging shall be regular in the sense that it is an identity mapping for a point estimate $\hat{\boldsymbol{\theta}}$; for then the definition of the variance-covariance matrix will include the conventional definition. There is a further point to watch. Suppose equations (1) are soluble in the ordinary sense except for $j = 1$. If we now estimate θ_1 by the foregoing general technique and substitute this solution into the remaining equations, then some or all of these remaining equations may no longer be soluble. We may expect this to happen when the estimate lies on the frontier of parameter space; and we shall have an instance of it when we come to consider the data for the Alpine Swifts.

A stochastic model for capture-recapture analysis

Let $p_i = 1 - q_i$ denote the probability that a particular individual I_j, known to be alive on occasion T_i, is captured (or recaptured) on that occasion. Since (by hypothesis) this probability is constant for all individuals at risk of capture, p_i is an index of the experimenter's catching efficiency on the occasion T_i. Let $\phi_{ij} = 1 - \psi_{ij}$ denote the probability that the individual I_j is alive on the occasion T_{i+1}, given that it is alive on occasion T_i. Let μ_{ij} denote the probability that the individual I_j is recaught on some occasion subsequent to T_i, given that it is captured (or recaptured) on occasion T_i. Then we have the fundamental difference equation

$$\mu_{ij} = \phi_{ij}(p_{i+1} + q_{i+1}\mu_{i+1,j}); \quad \mu_{mj} = 0. \tag{2}$$

The boundary condition $\mu_{mj} = 0$ holds because the experimenter makes no further catches after the final occasion T_m. For a proof of the difference equation itself, suppose we know that I_j has been caught on occasion T_i. Then we know it is alive on occasion T_i. If it is recaught, it must certainly survive until occasion T_{i+1} at least, of which event there is probability ϕ_{ij}; and then it must either be recaught on occasion T_{i+1} given that it is then alive

(with probability p_{i+1}), or it must escape capture on occasion T_{i+1} (with probability q_{i+1}) and then be captured on some subsequent occasion given that it is alive at T_{i+1} (and the probability of this is $\mu_{i+1,j}$). The 'either-or' of the previous sentence is exclusive; and (2) follows at once. Now from (2)

$$\phi_{ij} = \phi_{ij}(p_{i+1}+q_{i+1}) \geqslant \phi_{ij}(p_{i+1}+q_{i+1}\mu_{i+1,j}) = \mu_{ij}$$

and

$$\mu_{ij} = \phi_{ij}\mu_{i+1,j}+\phi_{ij}p_{i+1}(1-\mu_{i+1,j}) \geqslant \phi_{ij}\mu_{i+1,j},$$

because the various probabilities lie between 0 and 1 inclusive. By putting $p_{i+1} = 1$ and 0 respectively in these inequalities we deduce that

$$\phi_{ij} \geqslant \mu_{ij} \geqslant \phi_{ij}\mu_{i+1,j} \tag{3}$$

is the best possible set of inequalities operating upon the μ's. Hence (3) defines the parameter space $\boldsymbol{\mu}$. The parameter space $\boldsymbol{\phi}$ is evidently the closed unit hypercube. From (2) we also have

$$\mu_{i-1,j}-\phi_{i-1,j} = \phi_{i-1,j}(p_i-1+q_i\mu_{ij}) = \phi_{i-1,j}q_i(-1+\mu_{ij});$$

and so

$$\phi_{i-1,j}q_i = \frac{\phi_{i-1,j}-\mu_{i-1,j}}{1-\mu_{ij}} \quad (i = 2, 3, \ldots, m). \tag{4}$$

Also

$$\frac{p_i}{q_i} = \frac{\phi_{i-1,j}-\phi_{i-1,j}q_i}{\phi_{i-1,j}q_i} = \frac{\phi_{i-1,j}(1-\mu_{ij})-(\phi_{i-1,j}-\mu_{i-1,j})}{\phi_{i-1,j}-\mu_{i-1,j}}$$

$$= \frac{\mu_{i-1,j}-\phi_{i-1,j}\mu_{ij}}{\phi_{i-1,j}-\mu_{i-1\,j}} \quad (i = 2, 3, \ldots, m). \tag{5}$$

Suppose I_j is captured on occasions $T_\alpha, T_\beta, \ldots, T_\theta$ and on no other occasions, and that we consider only information upon I_j on occasions from T_α onwards. The likelihood is then

$$L_j = \phi_{\alpha j}\phi_{\alpha+1,j}\cdots\phi_{\theta-1,j}p_\alpha p_\beta\cdots p_\theta\frac{q_\alpha q_{\alpha+1}\cdots q_\theta}{q_\alpha q_\beta\cdots q_\theta}(1-\mu_{\theta j}). \tag{6}$$

The likelihood of **D** will now be

$$L = L_1 L_2 \ldots L_n. \tag{7}$$

If we wish to take into account the possibility that I_j has remained uncaught in the population from some unknown instant of entry $K(j)$, then L_j must be multiplied by

$$\phi_{K(j),j}\phi_{K(j)+1,j}\cdots\phi_{\alpha-1,j}q_{K(j)}q_{K(j)+1}\cdots q_{\alpha-1},$$

in which $K(j)$ is a parameter to be estimated (or a random variable to be integrated out).

The maximum-likelihood solution results from choosing, subject to (3), the available parameters so that L is as large as possible. Before we can do this, however, we must by some assumption reduce the number of unknown parameters; for at present there are more unknown parameters than observations (i.e. elements of **D**), and hence the parameters cannot be uniquely determined. The analysis will, moreover, not be tractable from the numerical point of view unless we keep the number of unknown parameters fairly small (say less than 100). Thus we might assume that all individuals were equally healthy. At the same time, also for numerical convenience, we may employ the theorem stated in the introduction. According to this we may transform the parameter space before maximizing (7), and then, after estimates have been found in this transformed space, we can transform them back into the original space. We shall adopt this technique in the illustrations on Alpine Swifts by employing (4) and (5) to carry **p**-space into $\boldsymbol{\mu}$-space. Another technique worth considering is the deliberate use of an excess number of parameters, by the introduction of

new parameters which depend in a functionally determinate manner upon the minimal set of parameters assumed in formulating the model. In the augmented parameter space the minimal set will be some hypersurface. We may then maximize (7) upon this hypersurface by the use of Lagrangian undetermined multipliers. This will prove a convenient procedure when it is easier to work with many simple equations than with a smaller number of complicated ones.

The parameters which are sought are the ϕ's and the p's. The former specify death-rate, and the latter lead immediately to estimates of population size.

An experiment with Alpine Swifts

We shall analyse some data collected upon two colonies of Alpine Swifts (*Apus melba*) at Solothurn in Switzerland from 1920 to 1950.

A bird aged x years is called a nestling if $x < 1$, young if $1 \leqslant x < 2$, and adult if $x \geqslant 2$. An ornithologist can tell the difference between these three categories upon inspection. Alpine Swifts migrate annually. If a bird returns to the place of catching when it is adult, it is ornithologically reasonable to suppose that it will thereafter continue to return to this same place each year until it dies. This does not necessarily hold for nestlings or young. When a bird is caught for the first time a ring is placed upon its leg. All rings bear distinct numbers. Only a small number of birds are caught when young; and it is convenient, in view of the anomalous migration of young, to suppress all records of capturing young. A bird is said to be 'ringed' when it is caught for the first time in the *unsuppressed* records. We shall suppose that an adult bird has a constant probability $\phi = 1 - \psi$ of surviving one year. This is a much simpler assumption than that of the previous general treatment. It is not an unfair supposition in view of the fact that most birds die due to natural hazards rather than through old age. These hazards vary from year to year, for meteorological reasons amongst others; but the estimate, which is wanted, is naturally a hypothetical one referring to an 'average' bird in an 'average' year. Both these last two uses of the word 'average' are defined by the assumption that ϕ is to be regarded as a constant; that is to say, the death-rate for Alpine Swifts is the metric upon which an average is taken rather than, say, the mean winter temperatures or the supply of summer food. Since ϕ no longer depends upon a suffix j we may write μ_i for the probability that an adult bird (known to be alive in year i) is subsequently recaught. To deal with the anomalous behaviour of nestlings and young, we write π for the probability that a nestling (ringed at birth) survives *and* returns to the place of catching *two* years later. Define $\omega = \pi/\phi$.

We can now write down the probabilities for the various possible records on a single bird:

[1] Bird ringed in the unsuppressed records as an adult in year i and not recaught:

$$L_j = (1 - \mu_i). \tag{8}$$

[2] Bird ringed in the unsuppressed records as an adult in year i and recaught in years $\alpha, \beta, \ldots, \theta$ and in no other years:

$$L_j = \phi^{\theta - i} p_\alpha p_\beta \ldots p_\theta \frac{q_{i+1} q_{i+2} \ldots q_\theta}{q_\alpha q_\beta \ldots q_\theta} (1 - \mu_\theta). \tag{9}$$

[3] Bird ringed as a nestling in year $i - 1$ and not recaught in the unsuppressed records:

$$\begin{aligned} L_j = (1 - \pi) + \pi q_{i+1}(1 - \mu_{i+1}) &= 1 - \phi\omega + \phi\omega(1 - p_{i+1} - q_{i+1}\mu_{i+1}) \\ &= 1 - \omega\mu_i, \end{aligned} \tag{10}$$

by virtue of (2).

[4] Bird ringed as a nestling in year $i-1$ and in the unsuppressed records recaught in years $\alpha, \beta, \ldots, \theta$ and in no other years:

$$L_j = \text{expression (9) multiplied by } \omega. \tag{11}$$

[5] Bird found dead in year θ: in expression (9) we must replace $\phi(1-\mu_\theta)$ by ψ. Hence it is enough to introduce the factor

$$\frac{1-\phi}{\phi(1-\mu_\theta)}. \tag{12}$$

[6] Bird killed artificially for experimental purposes in year θ: introduce the factor

$$\frac{1}{1-\mu_\theta}. \tag{13}$$

To simplify the ensuing numerical analysis we now transform from the parameter space $(\phi, \pi, \mathbf{p})$ to the space $(\phi, \omega, \boldsymbol{\mu})$. This is justified by the theorem quoted in the introduction and effected by substituting (4) and (5) into (9) to yield

$$\begin{aligned}
&\phi^{\theta-i} p_\alpha p_\beta \ldots p_\theta \frac{q_{i+1} q_{i+2} \ldots q_\theta}{q_\alpha q_\beta \ldots q_\theta}(1-\mu_\theta) \\
&= \left(\frac{\phi-\mu_i}{1-\mu_{i+1}}\right)\left(\frac{\phi-\mu_{i+1}}{1-\mu_{i+2}}\right)\ldots\left(\frac{\phi-\mu_{\theta-1}}{1-\mu_\theta}\right)(1-\mu_\theta)\left(\frac{\mu_{\alpha-1}-\phi\mu_\alpha}{\phi-\mu_{\alpha-1}}\right)\left(\frac{\mu_{\beta-1}-\phi\mu_\beta}{\phi-\mu_{\beta-1}}\right)\ldots\left(\frac{\mu_{\theta-1}-\phi\mu_\theta}{\phi-\mu_{\theta-1}}\right) \\
&= (\phi-\mu_i)\left(\frac{\phi-\mu_{i+1}}{1-\mu_{i+1}}\right)\left(\frac{\phi-\mu_{i+2}}{1-\mu_{i+2}}\right)\ldots\left(\frac{\phi-\mu_{\theta-1}}{1-\mu_{\theta-1}}\right)\left(\frac{\mu_{\alpha-1}-\phi\mu_\alpha}{\phi-\mu_{\alpha-1}}\right)\left(\frac{\mu_{\beta-1}-\phi\mu_\beta}{\phi-\mu_{\beta-1}}\right)\ldots\left(\frac{\mu_{\theta-1}-\phi\mu_\theta}{\phi-\mu_{\theta-1}}\right).
\end{aligned} \tag{14}$$

Upon noting that $\mu_m = 0$ and collecting up expressions (8)–(14) inclusive, we find for the likelihood of the whole set of data $\mathbf{D}$,

$$L = \omega^A \left(\frac{1-\phi}{\phi}\right)^F \prod_{i=1}^{m-1} \{(1-\omega\mu_i)^{B_i}(1-\mu_i)^{C_i}(\phi-\mu_i)^{D_i}(\mu_i-\phi\mu_{i+1})^{E_i}\}, \tag{15}$$

where in the unsuppressed records

A = number of nestlings which are recaptured;

B_i = number of nestlings ringed in year $i-1$ which are never recaptured;

C_i = (number of adults ringed in year i and never recaptured)
- (number of adults ringed before year i and recaptured after year i)
- (number of nestlings ringed before year $i-1$ and recaptured after year i)
- (number of birds killed artificially or recaptured dead in year i);

D_i = number of birds which, having been ringed *either* as adults before year $i+1$ *or* as young before year i, were recaptured sometime after year $i+1$ but which were not recaptured in year $i+1$;

E_i = number of birds recaptured in year $i+1$;

F = number of birds found dead but not artificially killed.

It it thus seen that, under suitable assumptions; it is not necessary to know individual histories.

Let us renumber the suffices i so that they run over the values $i = 20, 21, \ldots, 50$ corresponding to the years 1920–50 inclusive. The data then yields $A = 205$, $F = 6$ and

i	B_i	C_i	D_i	E_i
20	0	+ 1	2	0
21	17	0	2	0
22	18	− 2	6	1
23	0	− 5	4	4
24	12	0	7	7
25	33	+ 2	3	7
26	25	− 5	11	0
27	12	0	11	0
28	35	− 11	17	0
29	25	− 17	17	7
30	0	− 21	15	7
31	27	− 17	10	18
32	24	− 23	18	18
33	42	− 29	28	25
34	60	− 37	31	28
35	82	− 51	34	42
36	100	− 51	26	58
37	129	− 15	44	16
38	103	− 41	51	9
39	82	− 51	57	3
40	129	− 58	47	17
41	178	− 57	41	21
42	125	− 53	36	37
43	175	− 44	60	23
44	113	− 65	51	33
45	132	− 66	60	43
46	155	− 90	61	61
47	189	− 96	59	81
48	216	−103	42	87
49	3	− 58	0	84
50	328	+ 18	0	0

(16)

The summary (16) shows that the terms in L involving μ_{20} are

$$(1-\mu_{20})(\phi-\mu_{20})^2.$$

To maximize L we must therefore minimize μ_{20}. According to (3) we get

$$\mu_{20} = \phi\mu_{21}. \tag{17}$$

Now substitute (17) into (15) and pick out the terms in μ_{21}

$$(1-\omega\mu_{21})^{17}(\phi-\mu_{21})^2(1-\phi\mu_{21})(\phi-\phi\mu_{21})^2.$$

Once again use (3) to obtain from (17)

$$\mu_{20} = \phi\mu_{21} = \phi^2\mu_{22}. \tag{18}$$

We can now eliminate μ_{20} and μ_{21} from (15). The terms in μ_{22} become

$$\phi^4(1-\phi\omega\mu_{22})^{17}(1-\omega\mu_{22})^{18}(\phi-\mu_{22})^6(1-\phi\mu_{22})^2(1-\phi^2\mu_{22})(\mu_{22}-\phi\mu_{23}). \tag{19}$$

The analysis may proceed in a straightforward manner until we reach the next run of zeros in E_i. Treating these in a similar fashion we find

$$\mu_{26} = \phi\mu_{27} = \phi^2\mu_{28} = \phi^3\mu_{29}. \tag{20}$$

Upon elimination of μ_{26}, μ_{27} and μ_{28} from (15), the stage is set for logarithmic differentiation, and the estimation follows from the simultaneous solution, subject to (3), of

$$\frac{E_i}{\mu_i-\phi\mu_{i+1}} = \frac{\omega B_i}{1-\omega\mu_i} + \frac{C_i}{1-\mu_i} + \frac{D_i}{\phi-\mu_i} + \frac{\phi E_{i-1}}{\mu_{i-1}-\phi\mu_i} \quad (i = 23, 24, 30, 31, \ldots, 48, 49), \tag{21}$$

$$\frac{1}{\mu_{22}-\phi\mu_{23}} = \frac{18}{1-\omega\mu_{22}} + \frac{17\phi}{1-\phi\omega\mu_{22}} + \frac{6}{\phi-\mu_{22}} + \frac{2\phi}{1-\phi\mu_{22}} + \frac{\phi^2}{1-\phi^2\mu_{22}}, \tag{22}$$

$$\frac{7}{\mu_{25}-\phi^4\mu_{29}} = \frac{33}{1-\omega\mu_{25}} + \frac{3}{\phi-\mu_{25}} + \frac{2}{1-\mu_{25}} + \frac{7\phi}{\mu_{24}-\phi\mu_{25}}, \tag{23}$$

$$\begin{aligned}\frac{7}{\mu_{29}-\phi\mu_{30}} = {} & \frac{25\omega}{1-\omega\mu_{29}} + \frac{35\phi\omega}{1-\phi\omega\mu_{29}} + \frac{12\phi^2\omega}{1-\phi^2\omega\mu_{29}} + \frac{25\phi^3}{1-\phi^3\omega\mu_{29}} \\ & + \frac{17}{\phi-\mu_{29}} + \frac{11\phi^2}{1-\phi^2\mu_{29}} - \frac{5\phi^3}{1-\phi^3\mu_{29}} + \frac{7\phi^4}{\mu_{25}-\phi^4\mu_{29}},\end{aligned} \tag{24}$$

$$\mu_{50} = 0, \tag{25}$$

$$\begin{aligned}0 = \Omega \equiv \frac{\partial \log L}{\partial \omega} = {} & \frac{205}{\omega} + \sum_i{}' \frac{-B_i\mu_i}{1-\omega\mu_i} \\ & - \frac{17\phi\mu_{22}}{1-\phi\omega\mu_{22}} - \frac{18\mu_{22}}{1-\omega\mu_{22}} - \frac{33\mu_{25}}{1-\omega\mu_{25}} \\ & - \frac{25\phi^3\mu_{29}}{1-\phi^3\omega\mu_{29}} - \frac{12\phi^2\mu_{29}}{1-\phi^2\omega\mu_{29}} - \frac{35\phi\mu_{29}}{1-\phi\omega\mu_{29}} - \frac{25\mu_{29}}{1-\omega\mu_{29}},\end{aligned} \tag{26}$$

$$\begin{aligned}0 = \Phi \equiv \frac{\partial \log L}{\partial \phi} = {} & \sum_i{}' \left\{ \frac{D_i}{\phi-\mu_i} - \frac{E_i\mu_{i+1}}{\mu_i-\phi\mu_{i+1}} \right\} + \frac{37}{\phi} - \frac{6}{1-\phi} \\ & - \frac{17\omega\mu_{22}}{1-\phi\omega\mu_{22}} + \frac{6}{\phi-\mu_{22}} - \frac{2\mu_{22}}{1-\phi\mu_{22}} - \frac{2\phi\mu_{22}}{1-\phi^2\mu_{22}} \\ & + \frac{3}{\phi-\mu_{25}} - \frac{28\phi^3\mu_{29}}{\mu_{25}-\phi^4\mu_{29}} - \frac{\mu_{23}}{\mu_{22}-\phi\mu_{23}} - \frac{35\omega\mu_{29}}{1-\phi\omega\mu_{29}} \\ & - \frac{24\phi\omega\mu_{29}}{1-\phi^2\omega\mu_{29}} - \frac{75\phi^2\omega\mu_{29}}{1-\phi^3\omega\mu_{29}} + \frac{17}{\phi-\mu_{29}} \\ & - \frac{22\phi\mu_{29}}{1-\phi^2\mu_{29}} + \frac{15\phi^2\mu_{29}}{1-\phi^3\mu_{29}} - \frac{7\mu_{30}}{\mu_{29}-\phi\mu_{30}}.\end{aligned} \tag{27}$$

In equations (26) and (27), $\sum_i{}'$ denotes summation over $i = 23, 24, 30, 31, \ldots, 48, 49$. The method of solving the foregoing set of 28 simultaneous equations will be discussed in the next section.

The next stage is the determination of the inverse variance-covariance matrix for $(\phi, \omega, \boldsymbol{\mu})$. As already mentioned in the introduction, it is not clear what one should do with co-ordinates on the frontier of the parameter space. In default of a better procedure, I have regarded (18) and (20) as algebraic identities; and thus taken

$$p_{21} = p_{22} = p_{27} = p_{28} = p_{29} = 0$$

as quantities not subject to sampling error. It then remains to calculate

$$c_{ij} = \frac{\partial}{\partial \mu_i}\left(\frac{1}{L}\frac{\partial L}{\partial \mu_j}\right) \quad (i,j = 22, 23, 24, 25, 29, 30, \ldots, 48, 49);$$

$$c_{i\phi} = \frac{\partial}{\partial \mu_i}\left(\frac{1}{L}\frac{\partial L}{\partial \phi}\right), \quad c_{i\omega} = \frac{\partial}{\partial \mu_i}\left(\frac{1}{L}\frac{\partial L}{\partial \omega}\right) \quad (i = 22, 23, 24, 25, 29, 30, \ldots, 48, 49);$$

$$c_{\phi\omega} = \frac{\partial}{\partial \phi}\left(\frac{1}{L}\frac{\partial L}{\partial \omega}\right), \quad c_{\phi\phi} = \frac{\partial}{\partial \phi}\left(\frac{1}{L}\frac{\partial L}{\partial \phi}\right), \quad c_{\omega\omega} = \frac{\partial}{\partial \omega}\left(\frac{1}{L}\frac{\partial L}{\partial \omega}\right).$$

The algebraic expressions for these quantities are lengthy but quite easily obtained from L, and therefore will not be printed here. The only non-vanishing c's are c_{ii}, $c_{\phi\phi}$, $c_{\omega\omega}$, $c_{\omega\phi} = c_{\phi\omega}$, $c_{i,i+1} = c_{i+1,i}$, $c_{i\omega} = c_{\omega i}$, $c_{i\phi} = c_{\phi i}$.

If we write

$$\mathbf{C} = \begin{pmatrix} c_{ij} & \vdots & c_{i\omega} & c_{i\phi} \\ \cdots & \vdots & \cdots & \cdots \\ c_{\omega j} & \vdots & c_{\omega\omega} & c_{\omega\phi} \\ c_{\phi j} & \vdots & c_{\phi\omega} & c_{\phi\phi} \end{pmatrix} \quad (i,j = 22, 23, 24, 25, 29, 30, \ldots, 49);$$

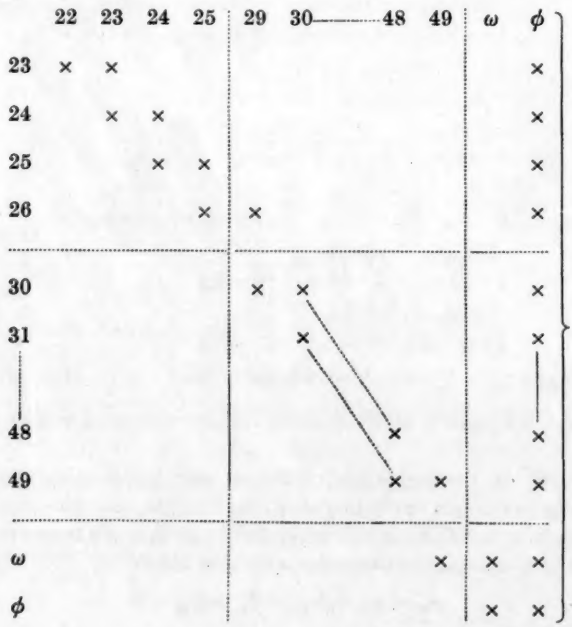
then $-\mathbf{C}^{-1}$ is the variance-covariance matrix of the estimates in the parameter space $(\boldsymbol{\mu}, \omega, \phi)$. To transform to the $(\mathbf{p}, \omega, \phi)$ space we take a first variation of (4) and obtain

$$\delta q_i = -\frac{1}{\phi(1-\mu_i)}\delta\mu_{i-1} + \frac{(\phi-\mu_{i-1})}{\phi(1-\mu_i)^2}\delta\mu_i + \frac{\mu_{i-1}}{\phi^2(1-\mu_i)}\delta\phi,$$

with the exception

$$\delta q_{26} = -\frac{1}{\phi(1-\phi^3\mu_{29})}\delta\mu_{25} + \frac{\phi^2(\phi-\mu_{25})}{(1-\phi^3\mu_{29})}\delta\mu_{29} + \left\{\frac{\mu_{25}}{\phi^2(1-\phi^3\mu_{29})} + \frac{3\phi\mu_{29}(\phi-\mu_{25})}{(1-\phi^3\mu_{29})^2}\right\}\delta\phi.$$

Consider now the 26×27 matrix $\mathbf{U}$ whose rows and columns are indexed according to the following scheme:



(28)

and whose elements vanish except in the last column and on the leading diagonal and super-diagonal. These remaining elements are specified by

$$u_{i+1,i} = -1/\phi(1-\mu_i) \quad (i = 22, 23, 24, 29, 30, \ldots, 47, 48);$$

$$\left.\begin{aligned} u_{ii} &= (\phi - \mu_{i-1})/\phi(1-\mu_i)^2 \\ u_{i\phi} &= \mu_{i-1}/\phi^2(1-\mu_i) \end{aligned}\right\} \quad (i = 23, 24, 25, 30, 31, \ldots, 48, 49);$$

$$u_{26,25} = -1/\phi(1-\phi^3\mu_{29});$$

$$u_{26,29} = \phi^2(\phi - \mu_{25})/(1-\phi^3\mu_{29})^2;$$

$$u_{26,\phi} = \frac{\mu_{25}}{\phi^2(1-\phi^3\mu_{29})} + \frac{3\phi\mu_{29}(\phi-\mu_{25})}{(1-\phi^3\mu_{29})^2};$$

$$u_{\omega\omega} = u_{\phi\phi} = 1; \quad u_{\omega,49} = u_{\omega\phi} = u_{\phi\omega} = 0.$$

Then the variance-covariance matrix for estimates in the $(\mathbf{p}, \omega, \phi)$ space is $-\mathbf{U}\mathbf{C}^{-1}\mathbf{U}'$, whose rows and columns are indexed in the same fashion as the rows of $\mathbf{U}$ in (28).

The estimates of $\mathbf{p}$ themselves follow from (4).

NUMERICAL ANALYSIS

The main numerical labour lies in the solution of the 28 simultaneous equations (21)–(27) inclusive. This was done by iteration as follows:

(i) Choose trial values of ϕ, ω and μ_{22}. Then use equations (21)–(24) inclusive to calculate in succession $\mu_{23}, \mu_{24}, \mu_{25}, \mu_{29}, \mu_{30}, \ldots, \mu_{49}, \mu_{50}$, this last value of μ_{50} coming from (21) with $i = 49$. This sequence of calculations will be called a minor cycle. The final value of μ_{50} so obtained should be zero to agree with (25); but in fact it will not be, due to the inexact trial choice of μ_{22}. So keeping ϕ, ω both fixed, adjust the trial value of μ_{22} and begin a new minor cycle. Repeat the minor cycles with successive adjustments of μ_{22} until the resulting value of μ_{50} satisfies (25).

Then

(ii) use equations (26) and (27) to calculate Ω and Φ. The work so far covered in (i) and (ii) will be called a major cycle.

(iii) Do two more major cycles with fresh trial values of ϕ, ω. We now have three pairs of values of Ω, Φ. Carry out a bivariate inverse interpolation in the (Ω, Φ)-plane to estimate values of ϕ, ω such that $\Omega = \Phi = 0$.

(iv) Finally, repeat the major cycles, with successive approximations upon ϕ, ω until (26) and (27) are satisfied.

In carrying out the minor cycles we have to observe (3). Thus any minor cycle can be stopped, when only partly completed, as soon as (3) is violated by a poor choice of μ_{22}. This helps to shorten the work considerably. Increasing μ_{22} increases all the other μ's for fixed ϕ, ω; and decreasing μ_{22} decreases all the other μ's. The latter values of μ are extremely sensitive to small changes in μ_{22}; in the final solution a change of 10^{-10} in μ_{22} gave a change of 4×10^{-5} in μ_{50}. Similarly, in the major cycles Ω and Φ were very sensitive to changes in μ_{50}. On that account the work was carried out retaining 10 significant figures throughout, with a number of interpolating refinements built into the final stages. After the final major cycle, the computing errors in ϕ and ω were determined variationally and found to be 4×10^{-6}

and 1×10^{-6} respectively. This leaves sufficient margin in view of the standard errors of these quantities.

All the foregoing work was done on ordinary desk calculators. A minor cycle could be completed in about a day; and a major cycle in about 10 days.

The 27×27 matrix $\mathbf{C}$ was evaluated on a desk calculator. $\mathbf{C}$ was inverted on SEAC. This required a couple of days for coding, followed by about 4 hr. machine time. In view of the large proportion of zeros in C, however, this inversion could probably have been accomplished with about two months' work on a desk calculator. Finally, the standard errors were calculated from the diagonal elements of $-\mathbf{UC}^{-1}\mathbf{U}'$ on a desk calculator.

Numerical results

The final major cycle employed the values

$$\phi = 0{\cdot}82177\,252, \quad \omega = 0{\cdot}15231\,687.$$

s	μ
22	0·49114 70078
23	0·55283 78505
24	0·48551 39880
25	0·39790 15014
29	0·57753 15026
30	0·62797 49077
31	0·68978 94273
32	0·65635 69119
33	0·64690 45524
34	0·63783 00155
35	0·63643 78806
36	0·57590 81349
37	0·37055 55369
38	0·37884 51461
39	0·42295 19018
40	0·50268 84610
41	0·54322 89445
42	0·57591 99689
43	0·54504 16713
44	0·59087 04335
45	0·61829 34809
46	0·64147 85385
47	0·63248 97872
48	0·57021 21461
49	0·40731 92325
50	0·00000 00000

these values being interpolated between the independent minor cycles

$$\mu_{22} = 0{\cdot}49114\,70079 \text{ leading to } \mu_{50} = +0{\cdot}00002\,70661,$$

$$\mu_{22} = 0{\cdot}49114\,70078 \text{ leading to } \mu_{50} = -0{\cdot}00001\,02407.$$

For a reason to be noticed presently, we classify all birds as either X-birds or Y-birds. A bird is an X-bird if and only if it is a nestling when ringed, and is a Y-bird if and only if it is an adult when ringed. Since ringing, by definition, refers only to the unsuppressed records, these two classes are mutually exclusive and exhaustive. The following table summarizes

for each year the number of birds captured, together with estimates of the experimenter's catching efficiency and of the current population size:

Year	Number of birds caught			Catching efficiency ($p_i \pm$ standard error)	Population size ($\pm$ standard error)
	X-birds	Y-birds	All birds		
1920	0	3	3	—	—
1921	17	2	19	0	—
1922	23	0	23	0	—
1923	2	2	4	$0{\cdot}100 \pm 0{\cdot}096$	40 ± 38
1924	19	11	30	$0{\cdot}364 \pm 0{\cdot}154$	82 ± 35
1925	36	16	52	$0{\cdot}320 \pm 0{\cdot}105$	162 ± 53
1926	32	6	38	$0{\cdot}241 \pm 0{\cdot}084$	158 ± 55
1927	12	11	23	0	—
1928	37	4	41	0	—
1929	33	0	33	0	—
1930	7	3	10	$0{\cdot}201 \pm 0{\cdot}071$	50 ± 18
1931	38	7	45	$0{\cdot}240 \pm 0{\cdot}082$	188 ± 64
1932	38	17	55	$0{\cdot}533 \pm 0{\cdot}094$	103 ± 18
1933	64	20	84	$0{\cdot}430 \pm 0{\cdot}080$	195 ± 36
1934	82	25	107	$0{\cdot}412 \pm 0{\cdot}065$	260 ± 41
1935	113	22	135	$0{\cdot}384 \pm 0{\cdot}060$	351 ± 54
1936	141	35	176	$0{\cdot}468 \pm 0{\cdot}056$	376 ± 45
1937	170	64	234	$0{\cdot}525 \pm 0{\cdot}052$	446 ± 45
1938	121	17	138	$0{\cdot}116 \pm 0{\cdot}028$	1190 ± 285
1939	91	9	100	$0{\cdot}066 \pm 0{\cdot}022$	1515 ± 494
1940	136	2	138	$0{\cdot}024 \pm 0{\cdot}014$	5726 ± 3279
1941	202	8	210	$0{\cdot}150 \pm 0{\cdot}035$	1401 ± 326
1942	143	25	168	$0{\cdot}201 \pm 0{\cdot}041$	837 ± 173
1943	213	42	255	$0{\cdot}342 \pm 0{\cdot}050$	745 ± 109
1944	139	16	155	$0{\cdot}177 \pm 0{\cdot}035$	876 ± 172
1945	172	30	202	$0{\cdot}264 \pm 0{\cdot}042$	765 ± 121
1946	191	39	230	$0{\cdot}309 \pm 0{\cdot}041$	743 ± 100
1947	236	60	296	$0{\cdot}403 \pm 0{\cdot}043$	734 ± 78
1948	271	58	329	$0{\cdot}464 \pm 0{\cdot}041$	709 ± 63
1949	47	72	119	$0{\cdot}484 \pm 0{\cdot}043$	246 ± 22
1950	285	55	340	—	—

The expectation of life of a bird is $1/(1-\phi)$ years, and this, together with its standard error, is $5{\cdot}61 \pm 0{\cdot}30$ years.

The probability that a bird will survive 2 years from birth and then return to its place of birth is $\phi\omega$, and this, together with its standard error, is $0{\cdot}1252 \pm 0{\cdot}0088$. Thus about one nestling in eight returns two years after birth.

COMMENT ON THE DATA AND THEIR NUMERICAL RESULTS

The data were collected by H. Arn of Solothurn, who was primarily interested in quite different problems to those for which his data have now been used. In particular, because of his other interests, he made a deliberate attempt in some years to catch unringed adults and in other years to ring as many nestlings as possible. As a result the catching is not random, the various birds at risk have not always had equal chances of capture, and the fundamental assumption of the present capture-recapture method is not satisfied. The table above shows evidence of this lack of random catching. For example, in 1949 the experimenter

appears to have concentrated on capturing adult birds; and in this year he has secured an unusually large number of *Y*-birds in a fairly small total catch. As a result in 1949 his catching efficiency (based upon the false assumption that he had been catching at random) is artificially inflated, and therefore the estimate of population size is made much too low. The reverse effect takes place in the years 1939, 1940 and 1941 when he has evidently concentrated on ringing nestlings.

It was nevertheless decided to analyse the data as though they had been based upon random catching, because considerable ornithological interest attaches to the age of Alpine Swifts in that they enjoy an unusually long expectation of life in the wild state, and because the present data is quite unique in its extensiveness. Since the assumption of random catching is not here valid, the numerical results must be interpreted with a good deal of reserve. The individual year-by-year population sizes carry very little weight, although the general tendency of the size of the population to increase with time is borne out by other evidence not presented here. The probability of 1/8 for the return of a nestling two years later is also distinctly suspect. On the other hand, the estimate of expectation of life is in good accord with estimates obtained from other less extensive experiments, and is less liable to be incorrect although its standard error may well be wrong.

The fact that the numerical results are for these reasons suspect will concern the ornithologist: but the main object of the present paper is to provide a theoretical analysis of the general capture-recapture problem, and to show that the analysis can in fact be handled numerically. This has been demonstrated.

I am indebted to Herr H. Arn of Solothurn, who collected the original data upon Alpine Swifts, and to Dr D. Lack of the Edward Grey Institute of Field Ornithology, to whom the data were communicated and by whom they were transmitted to me. I am also indebted to the computing staff of the Lectureship in the Design and Analysis of Scientific Experiment, University of Oxford, for carrying out much of the numerical analysis. In particular, I am glad to thank Mrs M. Parke who carried the main burden of the work. I also wish to thank Dr J. Curtiss, Mr J. Todd, and the staff of the Computation Laboratory of the National Bureau of Standards, Washington, D.C., for the use of SEAC to invert the 27×27 reciprocal variance-covariance matrix. I should like to record helpful discussions on the subject of capture-recapture analysis with Dr D. J. Finney, Prof. R. A. Fisher, Mr P. H. Leslie, and Prof. P. A. P. Moran. Finally, I am indebted to the referee for useful comments.

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THE USE OF CHAIN-BINOMIALS WITH A VARIABLE CHANCE OF INFECTION FOR THE ANALYSIS OF INTRA-HOUSEHOLD EPIDEMICS

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1. Introduction

An important characteristic of highly infectious diseases is that they tend to produce groups of cases occurring within households rather than separate cases scattered throughout the community. If all the cases in a household were due to some single source of infection such as typhoid-infected water, then we should expect the total number of cases to follow a binomial distribution. On the other hand, if the disease were introduced into the household by one of its members and then transmitted from person to person, a very different situation would arise. For a disease involving a short period of high infectivity and an approximately constant incubation period, we should expect to be able to distinguish different generations of the intra-household epidemic. There might be a binomial distribution of secondary cases resulting from contact with the primary case, to be followed later by another binomial distribution of tertiary cases amongst the susceptibles who had previously escaped, and so on. In his classic paper on this subject Greenwood (1931) introduced such a chain-binomial model for the investigation of measles epidemics, for which the assumptions made are thought to be approximately true. When the period of infectivity is more extended, as with scarlet fever or whooping cough for example, the stochastic model recently discussed by Bailey (1953) may be more appropriate. Greenwood examined data on the 1926 measles epidemic in St Pancras, and showed that, so far as the total number of cases in a household was concerned, the hypothesis of a simple binomial distribution was quite inadequate, while the distribution expected from the chain-binomial model gave a satisfactory fit to the observations. It is important to notice, however, that this material provided information only on the *total* size of the epidemic in a given household; no analysis of the individual links of the chain was possible. Now Wilson, Bennett, Allen & Worcester (1939), in their investigation of cases of measles occurring in Providence, Rhode Island, during 1929–34, were able to go a stage further and break down the data into its constituent parts. Of course it must be recognized that departures from a constant incubation period, or the possibility of multiple primary cases, give rise to certain difficulties in the so-called chaining of this kind of material. However, it seemed likely that in the present case this source of confusion would not be serious. Wilson *et al.* were able to show that in none of the groups of available data did the Greenwood model give an adequate description of the numbers of cases occurring in the separate generations through which the epidemic passed, though it did sometimes give a satisfactory fit to the distribution of the total number of cases in a household. In a later paper, Wilson (1947) considered a somewhat different approach. According to this, in families containing two susceptibles in addition to the first case, for example, one could estimate separately the chance of one susceptible being attacked and the chance

that the other would be attacked given that the first had already become infected. This principle can be extended to families of any size, though it has the appreciable disadvantage that it is not clearly related to the epidemiological process known to be taking place. A further discussion of these problems was undertaken by Greenwood (1949). He made the important observation that the failure of the binomial distribution to represent the distribution of cases at stages in the chain where a goodness-of-fit is possible could be due to variations between households in the average chance of infection. Such a state of affairs might well be expected to result from natural variations in both heredity and environment. This remark of Greenwood's is taken as the starting point for the present paper, in which it is shown that a satisfactory mathematical model can be obtained, at least for the small amount of suitable data available, by retaining the hypothesis of a chain of binomial distributions, together with the additional assumption that the average chance of infection varies between different households according to a suitable β-distribution. This involves fitting two parameters instead of only one as previously. Another important modification of detail in Greenwood's original scheme has lately been suggested by Lidwell & Sommerville (1951). Greenwood (1931) assumed that the chance of any susceptible being infected was independent of the number of infectious persons to whom he is exposed. Lidwell & Sommerville, however, considered the alternative assumption that the risk of infection for any susceptible is the same with respect to each infectious individual. This of course affects only families with more than two susceptibles in addition to the primary case. In their analysis of the distributions of the total number of cases of the common cold in households, Lidwell & Sommerville showed that on the whole the Greenwood model was inadequate whereas their own modified scheme fitted quite well. For a useful discussion of the epidemiological aspects of the problem of estimating infectiousness Hope Simpson's (1952) recent paper may be consulted.

Households with only two individuals, i.e. containing only one susceptible in addition to the primary case, permit one to estimate the average chance of cross-infection, but are too small to exhibit a chain of cases. We shall therefore not consider these any further, but shall proceed to a discussion of households of three and four persons, for which there is observational material available for testing. The treatment of larger households is more complicated though similar in principle.

2. Analysis of households with three individuals

We will now consider households of three individuals, containing two susceptibles besides the primary case. Let us suppose that the probability of a susceptible contracting the disease on exposure to an infectious case is $p = 1-q$. Then the first generation of secondary cases, following the primary case, will take the values 0, 1 or 2, with frequencies q^2, $2pq$ and p^2. For the values 0 and 2 the epidemic within the household comes to an end; in the first case there is no further source of infection, and in the second there are no more susceptibles. On the other hand, a single secondary case may be followed in the second generation by 0 or 1 tertiary case, with frequencies q and p. The expected and observed numbers of households falling into the four classes are shown in Table 1, which also includes some suitable material taken from the Providence measles experience already referred to (Wilson *et al.* 1939), together with an obvious notation for the different kinds of chain. Wilson *et al.* analysed their material in several different ways, both including and excluding

children under 7 months or over 10 years old. Moreover, some of their tables are really relevant to families containing more than three children. We have decided for the present purpose to confine ourselves to the data on households of three, containing just two further susceptibles, apart from the primary case, at risk between the ages of 7 months and 10 years. This seemed to be the most reasonable way of selecting what might be expected to

Table 1. *Greenwood chain for households of three*

Type of chain	Expected no. of households	Observed no. of households	Providence measles data (see text)	Fitted values
$1 \to 0$	nq^2	a	34	14·9
$1 \to 1 \to 0$	$2npq^2$	b	25	23·5
$1 \to 1 \to 1 \to 0$	$2np^2q$	c	36	87·7
$1 \to 2 \to 0$	np^2	d	239	207·9
Total	n	n	334	334·0

be a fairly homogeneous group of data. In analysing their data Wilson *et al.* estimated the parameter p by equating the expected size of the *total* epidemic to the observed value. This is not necessarily very efficient. They then examined the fit to the four classes shown in Table 1. A more satisfactory procedure is to estimate p by maximum likelihood using all four classes shown in the table. It is easy to see that this efficient estimate is

$$\hat{p} = (b + 2c + 2d)/(2a + 3b + 3c + 2d), \tag{1}$$

with large sample variance $$\operatorname{var} \hat{p} = pq/2n(1 + pq). \tag{2}$$

Applying formula (1) to the data shown in Table 1, we find $p = 0{\cdot}789$. The corresponding expectations appear in the right-hand column. The goodness-of-fit χ^2 is 59·8, which for 2 degrees of freedom gives $P < 0{\cdot}001$. Thus the Greenwood model, which for groups of three is unaffected by the modification suggested by Lidwell & Sommerville, is quite inappropriate as it stands. Comparing the actual observations with their expectations in Table 1 shows that there are more families than expected, in which the epidemic either does not go beyond the primary case or else involves the remaining two susceptibles immediately. This fits in with Greenwood's (1949) suggestion that the chance of infection may vary from family to family. Suppose we now assume that this source of variation in p between households is represented by the β-distribution

$$dF = \frac{1}{B(x, y)} p^{x-1}(1-p)^{y-1}\, dp, \quad (0 \leqslant p \leqslant 1). \tag{3}$$

All we have to do is to average the expectations for each kind of chain over all households and exhibit them as functions of x and y. As the individual frequencies in Greenwood or Lidwell chains can always be represented in terms of quantities like $p^r q^s$, it is convenient to work out the expectation of this quantity. We have

$$\begin{aligned} Ep^r q^s &= \int_0^1 p^{x+r-1} q^{y+s-1}\, dp / B(x, y) \\ &= B(x + r, y + s)/B(x, y). \end{aligned}$$

Therefore

$$Ep^r q^s = x(x+1)\dots(x+r-1)\,y(y+1)\dots(y+s-1)/(x+y)(x+y+1)\dots(x+y+r+s-1). \quad (4)$$

In particular

$$Ep^r = x(x+1)\dots(x+r-1)/(x+y)(x+y+1)\dots(x+y+r-1), \quad (5)$$

and

$$Eq^s = y(y+1)\dots(y+s-1)/(x+y)(x+y+1)\dots(x+y+s-1). \quad (6)$$

If we apply these results to the simple expectations shown in Table 1 we obtain the derived set of values set out in Table 2, which we shall proceed to analyse by the familiar technique of maximum-likelihood scoring. The maximum-likelihood scores for x and y are clearly

$$\left.\begin{aligned} S_x &= \left(\frac{b+c+d}{x}+\frac{c+d}{x+1}\right)-\left(\frac{n}{x+y}+\frac{n}{x+y+1}+\frac{b+c}{x+y+2}\right),\\ \text{and}\quad S_y &= \left(\frac{a+b+c}{y}+\frac{a+b}{y+1}\right)-\left(\frac{n}{x+y}+\frac{n}{x+y+1}+\frac{b+c}{x+y+2}\right).\end{aligned}\right\} \quad (7)$$

Table 2. *Modified chain for household of three with variable p*

Type of chain	Expected no. of households	Observed no. of households	Providence measles data (see text)	Fitted values
$1\to 0$	$ny(y+1)/(x+y)(x+y+1)$	a	34	34·9
$1\to 1\to 0$	$2nxy(y+1)/(x+y)(x+y+1)(x+y+2)$	b	25	22·7
$1\to 1\to 1\to 0$	$2nyx(x+1)/(x+y)(x+y+1)(x+y+2)$	c	36	37·6
$1\to 2\to 0$	$nx(x+1)/(x+y)(x+y+1)$	d	239	238·8
Total	n	n	334	334·0

The expected amounts of information are

$$\left.\begin{aligned} I_{xx} &= \frac{n(x+2y+1)}{x(x+y)(x+y+1)}+\frac{nx(x+3y+2)}{(x+1)(x+y)(x+y+1)(x+y+2)}-J,\\ I_{xy} &= -J,\\ I_{yy} &= \frac{n(2x+y+1)}{y(x+y)(x+y+1)}+\frac{ny(3x+y+2)}{(y+1)(x+y)(x+y+1)(x+y+2)}-J,\\ \text{where}\quad J &= \frac{n}{(x+y)^2}+\frac{n}{(x+y+1)^2}+\frac{2nxy}{(x+y)(x+y+1)(x+y+2)^2}.\end{aligned}\right\} \quad (8)$$

In many cases it will be easier to use the observed amounts of information, which can be derived from the three different expressions in brackets making up the two scores in (7) simply by replacing the denominators by their squares.

An easy way to obtain two rough estimates of x and y to commence the iterative process is to amalgamate the second and third classes in Table 2, and then equate the observations to their expectations. This gives

$$\left.\begin{aligned} \check{x} &= (b+c)(b+c+2d)/\{4ad-(b+c)^2\},\\ \text{and}\quad \check{y} &= (b+c)(2a+b+c)/\{4ad-(b+c)^2\}.\end{aligned}\right\} \quad (9)$$

If we now use these results on the Providence measles data, we obtain first the initial estimates from equation (9), giving

$$\check{x} = 1{\cdot}142 \quad \text{and} \quad \check{y} = 0{\cdot}273,$$

for which values the scores are $S_x = -1{\cdot}1253$ and $S_y = +2{\cdot}1255$. The information and covariance matrices are respectively

$$\{I\} = \begin{bmatrix} 63{\cdot}8027 & -226{\cdot}696 \\ -226{\cdot}696 & 1075{\cdot}635 \end{bmatrix} \quad \text{and} \quad \{V\} = 10^{-2} \times \begin{bmatrix} 6{\cdot}2401 & +1{\cdot}3151 \\ +1{\cdot}3151 & 0{\cdot}3701 \end{bmatrix}. \tag{10}$$

The adjustments to be made to $\check{x}$ and $\check{y}$ are therefore $-0{\cdot}042$ and $-0{\cdot}007$ giving 1·100 and 0·266, respectively. A further stage of iteration makes a slight change to

$$\hat{x}_3 = 1{\cdot}092 \pm 0{\cdot}250 \quad \text{and} \quad \hat{y}_3 = 0{\cdot}264 \pm 0{\cdot}061, \tag{11}$$

where we have followed the estimates by their standard errors derived from the covariance matrix, and the subscripts refer to the size of the household. The expected values are given in the right-hand column of Table 2. The fit to the data is now very good indeed, the χ^2 with one degree of freedom being about 0·32. It is interesting to observe that since y is less than unity the β-distribution is J-shaped with an infinite ordinate at $p = 1$. This is in keeping with the excess of families observed in Table 1, in which both susceptibles succumbed immediately to infection by the primary case.

3. Analysis of households with four individuals

When we come on to households with a total of four susceptibles, i.e. three in addition to the primary case, then the modification introduced by Lidwell & Sommerville begins to have an effect. It influences the tertiary distribution, in which, when there are exactly two secondary cases, the values 0 and 1 will occur with probabilities q^2 and $1-q^2$ instead of q and p. Let us again use the Providence measles data, confining ourselves to households of four, containing just three further susceptibles, apart from the primary cases, at risk between the ages of 7 months and 10 years. The appropriate expectations and observations are all collected together in Table 3. The maximum likelihood estimate for the Greenwood model is plainly

$$\hat{p} = (b + 2c + 2d + 3e + 3f + 3g + 3h)/(3a + 5b + 6c + 4d + 6e + 5f + 4g + 3h). \tag{12}$$

Table 3. *Greenwood and Lidwell chains for households of four*

Type of chain	Expected no. of households		Observed no. of households	Providence data	Fitted values	
	Greenwood	Lidwell			Greenwood	Lidwell
$1 \to 0$	nq^3	nq^3	a	4	0·9	1·2
$1 \to 1 \to 0$	$3npq^4$	$3npq^4$	b	3	0·4	0·7
$1 \to 1 \to 1 \to 0$	$6np^2q^6$	$6np^2q^4$	c	1	0·7	1·0
$1 \to 2 \to 0$	$3np^2q^2$	$3np^2q^3$	d	8	8·2	2·2
$1 \to 1 \to 1 \to 1 \to 0$	$6np^3q^3$	$6np^3q^3$	e	4	2·7	3·4
$1 \to 1 \to 2 \to 0$	$3np^3q^2$	$3np^3q^3$	f	3	6·5	7·3
$1 \to 2 \to 1 \to 0$	$3np^3q$	$3np^3q(1-q^2)$	g	10	31·0	38·7
$1 \to 3 \to 0$	np^3	np^3	h	67	49·6	45·5
Total	n	n	n	100	100·0	100·0

Substituting the Providence data gives $p = 0{\cdot}791$, and the expected numbers are as shown in the table. The goodness-of-fit test, amalgamating the first four classes and then the next two, gives $P < 0{\cdot}001$. To estimate p on the Lidwell model we have to solve

$$S_p = \frac{b+2c+2d+3e+3f+2g+3h}{p} - \frac{3a+4b+4c+3d+3e+2f+g}{q} + \frac{2qg}{1-q^2} = 0. \quad (13)$$

Taking a few trial values and then interpolating inversely soon gives the required root. We find, for the data of Table 3, that $p = 0{\cdot}769$. Again a very poor fit is obtained with $P < 0{\cdot}001$.

Table 4. *Modified Greenwood chain for households of four with variable p*

Type of chain	Expected no. of households	Observed no. of households	Providence data	Fitted values
$1 \to 0$	$ny(y+1)(y+2)/(x+y)\dots(x+y+2)$	a	4	4·9
$1 \to 1 \to 0$	$3nxy(y+1)(y+2)(y+3)/(x+y)\dots(x+y+4)$	b	3	2·6
$1 \to 1 \to 1 \to 0$	$6nx(x+1)y(y+1)(y+2)(y+3)/(x+y)\dots(x+y+5)$	c	1	1·9
$1 \to 2 \to 0$	$3nx(x+1)y(y+1)/(x+y)\dots(x+y+3)$	d	8	5·0
$1 \to 1 \to 1 \to 1 \to 0$	$6nx(x+1)(x+2)y(y+1)(y+2)/(x+y)\dots(x+y+5)$	e	4	2·1
$1 \to 1 \to 2 \to 0$	$3nx(x+1)(x+2)y(y+1)/(x+y)\dots(x+y+4)$	f	3	3·0
$1 \to 2 \to 1 \to 0$	$3nx(x+1)(x+2)y/(x+y)\dots(x+y+3)$	g	10	13·3
$1 \to 3 \to 0$	$nx(x+1)(x+2)/(x+y)\dots(x+y+2)$	h	67	67·1
Total	n	n	100	99·9

We now turn to our previous assumption that p varies between households and has the β-distribution given by (3). Applying this modification to the expectations of the Greenwood model in a manner similar to that employed for households of size three, we obtain the expectations and observations shown in Table 4, which is analogous to Table 2. The maximum likelihood scores are

$$\left.\begin{aligned} S_x &= \left(\frac{n-a}{x} + \frac{n-a-b}{x+1} + \frac{e+f+g+h}{x+2}\right) \\ &\quad - \left(\frac{n}{x+y} + \frac{n}{x+y+1} + \frac{n}{x+y+2} + \frac{n-a-h}{x+y+3} + \frac{b+c+e+f}{x+y+4} + \frac{c+e}{x+y+5}\right), \\ S_y &= \left(\frac{n-h}{y} + \frac{n-g-h}{y+1} + \frac{a+b+c+e}{y+2} + \frac{b+c}{y+3}\right) - \left(\dots\right), \end{aligned}\right\} \quad (14)$$

where the second member of S_y in brackets is the same as the second member of S_x. The information functions are

$$\left.\begin{aligned} I_{xx} &= \left(\frac{n-a}{x^2} + \frac{n-a-b}{(x+1)^2} + \frac{e+f+g+h}{(x+2)^2}\right) - K, \\ I_{xy} &= -K, \\ I_{yy} &= \left(\frac{n-h}{y^2} + \frac{n-g-h}{(y+1)^2} + \frac{a+b+c+e}{(y+2)^2} + \frac{b+c}{(y+3)^2}\right) - K, \end{aligned}\right\} \quad (15)$$

where

$$K = \frac{n}{(x+y)^2} + \frac{n}{(x+y+1)^2} + \frac{n}{(x+y+2)^2} + \frac{n-a-h}{(x+y+3)^2} + \frac{b+c+e+f}{(x+y+4)^2} + \frac{c+e}{(x+y+5)^2}.$$

Some additional accuracy can no doubt be gained if the observed numbers in the several numerators in (15) are replaced by their expectations. There seems to be no obvious way of finding simple initial estimates, but with a few trial values it is not difficult to get sufficiently accurate approximations. Scoring with $x = 1{\cdot}43$ and $y = 0{\cdot}31$ gives

$$S_x = +0{\cdot}4124 \quad \text{and} \quad S_y = -0{\cdot}2782.$$

The information and covariance matrices are

$$\{I\} = \begin{bmatrix} 14{\cdot}6024 & -55{\cdot}233 \\ -55{\cdot}233 & 303{\cdot}810 \end{bmatrix} \quad \text{and} \quad \{V\} = 10^{-2} \times \begin{bmatrix} 21{\cdot}925 & +3{\cdot}9861 \\ +3{\cdot}9861 & 1{\cdot}0538 \end{bmatrix}. \tag{16}$$

The required adjustments to the estimates are therefore $+0{\cdot}08$ and $+0{\cdot}01$. We may therefore write the maximum-likelihood estimates as

$$\hat{x}_4 = 1{\cdot}51 \pm 0{\cdot}47 \quad \text{and} \quad \hat{y}_4 = 0{\cdot}32 \pm 0{\cdot}10. \tag{17}$$

Again, the expected values appearing in Table 4 show a very close fit to the observations. Grouping the first six classes into three pairs leaves 4 degrees of freedom altogether. The resulting goodness-of-fit χ^2 with 2 degrees of freedom is 2·20, a value which is entirely satisfactory. It hardly seems worth while at present to undertake a similarly modified treatment of the Lidwell chain model.

Comparing the results given in (11) with those in (17) shows no significant difference between the estimates obtained for households of three and households of four. We may therefore combine these estimates. The weighted averages, with weights equal to the reciprocals of the variances, are

$$\hat{x} = 1{\cdot}18 \pm 0{\cdot}22 \quad \text{and} \quad \hat{y} = 0{\cdot}279 \pm 0{\cdot}052. \tag{18}$$

The average chance of infection, $\bar{p}$, is given by putting $r = 1$ in (5). Thus we obtain

$$\bar{p} = x/(x+y) = 0{\cdot}81, \tag{19}$$

using the values appearing in (18). If in the future it proves possible to group data on the household distribution of the cases of a disease according to some such classifications as geographical area or social class, etc., it will be interesting to see whether p has a more circumscribed distribution within each of these groups.

4. SUMMARY

Greenwood (1931) was the first to introduce the chain-binomial model to describe the household distribution of cases of a disease with a short period of high infectivity and a constant incubation period. Greenwood obtained satisfactory goodness-of-fits to the St Pancras measles data, at least so far as the distribution of the total number of cases was concerned. Wilson *et al.* (1939) showed that on their measles data for Providence, Rhode Island, even if the fit to the distribution of the total number of cases was satisfactory, the theory was quite inadequate when the distributions of the separate generations of the disease within a household were examined. Later, Greenwood (1949) suggested that this might be due to variation in the chance of infection, p, between households. In the present paper the Providence data are re-examined on the assumption that p varies between households according to a suitable β-distribution. Two parameters then have to be estimated. p appears

to have a J-shaped distribution, with an infinite ordinate at $p = 1$. Two groups of the Providence data for households of three and four individuals were investigated, and in both cases excellent fits of the theory to the data were obtained.

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SPREAD OF DISEASES IN A RECTANGULAR PLANTATION WITH VACANCIES

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1. Introduction

This paper discusses the spread of diseases from plant to plant in a rectangular plantation where there may be some missing plants, which will be described as 'vacancies'. The problem is considered in two ways: (i) the degree of agglomeration of diseased plants on any one occasion, (ii) the spread of disease over a period of time. In the work which gave rise to this investigation the plants were hops, and the spread of the virus disease, nettlehead, over a period of five years was examined.

The methods used for the first aspect of the investigation, which may conveniently be called *spatial*, derive from various writers, principally Krishna Iyer. Historically, the first attempt to deal with this problem was that of Todd (1940), although previously Cochran (1936) had examined the linear case. Todd considered the distribution of pairs of adjacent plants, or points on a lattice, on the assumption that the underlying distribution was the binomial. The method of *doublets*, as he called it, has been followed in most of the subsequent work, although Finney (1947) showed empirically that Todd's formula seriously over-estimated the variance, because in an agglomeration doublets do not occur independently of one another. Van der Plank (1946) applied Cochran's and Todd's methods to virological studies, but without knowing of Finney's criticisms. Furthermore, he only considered doublets in one direction.

Moran (1948) considered the distribution of doublets for a general lattice, not specifically rectangular, and also found third and fourth moments in a particular case of the rectangular lattice. Krishna Iyer gave general solutions to the problem for a rectangular lattice in two papers; diagonal pairs were excluded in the 1950 paper, but considered in that of 1949.

Moran and Krishna Iyer both distinguished two cases: (i) where the probability of any plant being diseased is the same as that of any other plant being diseased and independent of the number of diseased plants, and (ii) where the number of diseased plants is known and interest attaches only to their distribution over the area. These cases are called *free* and *non-free* sampling respectively. Following the accepted notation, diseased plants are referred to here as *black* and healthy plants as *white*, and it is the distributions of both black-black (*BB*) joins and black-white (*BW*) joins that are of interest in practice.

Todd also considered sets of three and four neighbouring plants, called respectively *triplets* and *quadruplets*, but for these Finney (1947) showed that the variance would be under-estimated on the binomial assumption. Sukhatme (1949) considered triplets and stated that the inclusion of diagonals gave a more efficient test than the exclusion of diagonals, i.e. resulted in smaller coefficients of variation. He showed, further, that when diagonals were included the coefficients of variation for triplets were larger than those for doublets. In consequence of these results no further attention is here paid to triplets.

None of the above-mentioned writers has considered what happens when there are vacancies on the lattice, although Moran's general formula can be adapted to this case. In the present paper the approach to the problem has been to generalize Krishna Iyer's methods

for a rectangular lattice to make them applicable where there are vacancies rather than to particularize Moran's methods. The results must of course be the same, but Krishna Iyer's methods are the more elegant, for calculating both the probabilities of the number of joins of a given type and the total number of joins. (For this latter, the method of his 1949 paper has been preferred.) Both the cases, including diagonals and excluding diagonals, are considered, these being referred to, by analogy with chess, as *queen* and *rook* cases respectively. In the particular example given later in the paper the queen case was thought appropriate, but formulae are given here for the rook case as well, for completeness.

The methods used for the second aspect of the investigation, which may be called *temporal*, derive from the work of Barnard (1947) and Pearson (1947). Barnard distinguished three possible types of significance test for a 2×2 table—of which the second or 2×2 comparative trial is appropriate here—and Pearson obtained formulae for actually testing the significance of observed results. All that is done here on this aspect is to consider the relevant method of analysis and apply it.

2. Spatial aspect

Consider a rectangular lattice m by n with vacancies, where $m+n = a$, $m \times n = b$. Suppose that there are u vacancies in the interior of the lattice, v on the border and w at the corners, where $u+v+w = U$. Points that are not vacancies will be called *existent* points. In the free sampling case let the probabilities of getting black and white points be respectively p and q, and in the non-free sampling case let the numbers of black and white points be respectively n_1 and n_2. These two colours need not necessarily account for all the existent points in either case. Only the non-free sampling case will be considered in detail, this being slightly the more complicated and also the more likely to be of practical use. The appropriate substitutions for free sampling will be indicated. All the probabilities derived below assume the points to have been selected at random and their colours then examined.

In the non-free sampling case, then, the probability of any point being black is $\frac{n_1}{b-U}$ and of any point being white is $\frac{n_2}{b-U}$. Thus the probability of any join of two points being *BB* is

$$\frac{n_1(n_1-1)}{(b-U)(b-U-1)} \quad \text{or} \quad \frac{n_1!\,(b-U-2)!}{(n_1-2)!\,(b-U)!}. \tag{1}$$

Similarly the probability of a *BW* join is

$$\frac{2n_1n_2}{(b-U)(b-U-1)}. \tag{2}$$

The corresponding probabilities for free sampling are p^2 and $2pq$.

Further, two joins can arise in only two ways. Either they can come from three adjacent points, in which case they will be described as *dependent* pairs of joins, or they can come from four points in two discrete pairs, when they will be called *independent* pairs of joins. For non-free sampling the probability of obtaining three adjacent black points is

$$\frac{n_1!\,(b-U-3)!}{(n_1-3)!\,(b-U)!}, \tag{3}$$

and of obtaining four black points in two discrete pairs is

$$\frac{n_1!\,(b-U-4)!}{(n_1-4)!\,(b-U)!}. \tag{4}$$

For BW joins the position is slightly different. If three adjacent points are to make two dependent BW pairs the points must be either BWB or WBW. The sum of the probabilities of the different kinds of dependent BW pairs is thus

$$\frac{n_1(n_1-1)n_2+n_1n_2(n_2-1)}{(b-U)(b-U-1)(b-U-2)}=\frac{n_1n_2(n_1+n_2-2)}{(b-U)(b-U-1)(b-U-2)}. \tag{5}$$

The probability of two independent BW pairs is

$$\frac{4n_1!\,n_2!\,(b-U-4)!}{(n_1-2)!\,(n_2-2)!\,(b-U)!}. \tag{6}$$

The probabilities corresponding to (3)–(6) for free sampling are respectively p^3, p^4, $pq(p+q)$, $4p^2q^2$.

Now Krishna Iyer (1950) has shown that the rth factorial moment about zero for the distributions of both BB and BW joins is $r!$ times the sum of the expectations of the different ways of obtaining r joins of that type in the lattice. The argument employed in that paper (pp. 199–201) can be readily modified to show that the above statement holds for both rook and queen cases, even if there are vacancies.

All that is necessary, therefore, is to calculate the numbers of joins or sets of joins of a given type for the particular case under examination, and this is done below.

(*a*) *Queen case.* On the lattice there are, in all, 4 corner points, $2(a-4)$ border points and $b-2a+4$ interior points. Thus the total number of joins of existent points on the lattice is A, where

$$2A = 3.4+5.2(a-4)+8(b-2a+4)-2V,$$

V being a correction for vacancies, i.e.

$$A = 4b-3a+2-V. \tag{7}$$

Now each interior vacancy contributes 8 to V, each border vacancy 5 and each corner vacancy 3, except that two neighbouring vacancies reduce V by 1. Thus, if p is the number of pairs of vacancies

$$V = 8u+5v+3w-p. \tag{8}$$

If μ is the expected number of joins of a given type we have, for non-free sampling from the appropriate expression (1) or (2): for BB joins

$$\mu = \frac{An_1!\,(b-U-2)!}{(n_1-2)!\,(b-U)!}, \tag{9}$$

and for BW joins

$$\mu = \frac{2An_1n_2}{(b-U)(b-U-1)}, \tag{10}$$

where A is derived from (7) and (8). For free sampling p^2 and $2pq$ are substituted for (1) and (2) respectively.

The numbers of dependent and independent pairs of joins must now be considered. In the first case, where there are three adjacent points, two joins can arise in B ways, where

$$B = 3.4+10.2(a-4)+28(b-2a+4)-W,$$

W being a correction for vacancies, i.e.

$$B = 4(7b-9a+11)-W. \tag{11}$$

Now each interior vacancy contributes 28 to W directly, each border vacancy 10 and each corner vacancy 3. Also each interior existent point with one neighbouring vacancy contributes a further 7 to W, one with two neighbouring vacancies $7+6$ $(=13)$, one with three $7+6+5$, etc. Similarly, border existent points with one neighbouring vacancy contribute 4 to W, those with two $4+3$, etc. Finally, corner existent points with one neighbouring vacancy contribute 2 to W, etc. Let there be x_i interior existent points with i neighbouring vacancies, y_i border existent points with i neighbouring vacancies and z_i corner existent points with i neighbouring vacancies. Then

$$W = 28u+10v+3w+(7x_1+13x_2+18x_3+22x_4+25x_5+27x_6+28x_7+28x_8) + (4y_1+7y_2+9y_3+10y_4+10y_5)+(2z_1+3z_2+3z_3). \quad (12)$$

Since the total number of pairs of joins is $\frac{A(A-1)}{2}$, and pairs can only be dependent or independent, the number of independent pairs is $\frac{A(A-1)}{2}-B$.

For BB joins we thus have, from (3) and (4), that the second factorial moment is $\mu'_{[2]}$, where

$$\mu'_{[2]} = \frac{2Bn_1!\,(b-U-3)!}{(n_1-3)!\,(b-U)!} + \frac{[A(A-1)-2B]\,n_1!\,(b-U-4)!}{(n_1-4)!\,(b-U)!}.$$

Now if σ^2 is the variance, $$\sigma^2 = \mu'_{[2]}+\mu-\mu^2. \quad (13)$$

Hence $$\sigma^2 = \frac{2Bn_1!\,(b-U-3)!}{(n_1-3)!\,(b-U)!} + \frac{An_1!\,(b-U-2)!}{(n_1-2)!\,(b-U)!} + \frac{[A(A-1)-2B]\,n_1!\,(b-U-4)!}{(n_1-4)!\,(b-U)!} - \left[\frac{An_1!\,(b-U-2)!}{(n_1-2)!\,(b-U)!}\right]^2. \quad (14)$$

Similarly, we have for BW joins, by substituting in (13) from (5), (6) and (10),

$$\sigma^2 = \frac{2Bn_1n_2(n_1+n_2-2)}{(b-U)\,(b-U-1)\,(b-U-2)} + \frac{2An_1n_2}{(b-U)\,(b-U-1)} + \frac{4[A(A-1)-2B]\,n_1!\,n_2!\,(b-U-4)!}{(n_1-2)!\,(n_2-2)!\,(b-U)!} - \left[\frac{2An_1n_2}{(b-U)\,(b-U-1)}\right]^2. \quad (15)$$

The only change necessary for the free sampling case is the substitution of the appropriate probability expressions in the first and second factorial moments. For both free and non-free sampling Krishna Iyer has shown that the distributions tend to the normal form as the size of the lattice increases, if there are no vacancies. Thus, if the number of vacancies remains finite these distributions here will also tend to the normal in the limiting case. This will be true for both queen and rook cases.

(*b*) *Rook case.* The number of points on the lattice being the same as before, the total number of joins of existent points on the lattice is A', where

$$2A' = 2\,.\,4+3\,.\,2(a-4)+4(b-2a+4)-2V',$$

V' being a correction for vacancies, i.e.

$$A' = 2b-a-V'. \quad (16)$$

Now each interior vacancy contributes 4 to V', each border vacancy 3 and each corner vacancy 2, except that two neighbouring vacancies reduce V' by 1. (Here, and for the remainder of the section devoted to the rook case, two points are only regarded as neighbours

if they are next to one another on the rank or file.) Thus, if p' is the number of pairs of vacancies, p' being in general different from p as defined above, we have

$$V' = 4u + 3v + 2w - p'. \tag{17}$$

The expressions (9) and (10) in the queen case are now replaced by the following values for the mean number of joins: for BB joins

$$\mu = \frac{A' n_1! (b-U-2)!}{(n_1-2)! (b-U)!}, \tag{18}$$

and for BW joins

$$\mu = \frac{2A' n_1 n_2}{(b-U)(b-U-1)}, \tag{19}$$

A' being derived from (16) and (17). For free sampling p^2 and $2pq$ are again substituted for (1) and (2) respectively.

For the calculation of the second moment we again need to consider the number of ways of getting the two different types of pairs of joins. Now, three adjacent points can here be derived in B' ways, where

$$B' = 1 \,.\, 4 + 3 \,.\, 2(a-4) + 6(b-2a+4) - W',$$

W' being a correction for vacancies, i.e.

$$B' = 6b - 6a + 4 - W'. \tag{20}$$

Now each interior vacancy contributes 6 to W' directly, each border vacancy 3 and each corner vacancy 2. Also each interior existent point with one neighbouring vacancy contributes a further 3 to W', one with two neighbouring vacancies $3+2$, etc. Similarly, border existent points with one neighbouring vacancy contribute 2, and so on, and corner existent points with one or two neighbouring vacancies 1. Let there be x'_i interior existent points with i neighbouring vacancies, y'_i border existent points with i neighbouring vacancies and z'_i corner existent points with i neighbouring vacancies, where x'_i is not in general the same as x_i, etc. Thus

$$W' = 6u + 3v + 2w + (3x'_1 + 5x'_2 + 6x'_3 + 6x'_4) + (2y'_1 + 3y'_2 + 3y'_3) + (z'_1 + z'_2). \tag{21}$$

Since the total number of pairs of joins is $\frac{A'(A'-1)}{2}$, the number of independent pairs is, by the same reasoning as for the queen case, $\frac{A'(A'-1)}{2} - B'$.

Hence for BB joins we have, by substituting in (13) from (3), (4) and (9), the same expression as (14) with A' written for A and B' for B. Similarly, the variance of the number of BW joins is given by making the necessary modifications in (15). As before, the changes required for the free sampling case are merely in the probability expressions.

3. Applications

The method as outlined above is completely general, but its application can best be described by means of a specific example, for further details of which see the paper of Legg (1953). The apparent spread of hop nettlehead disease was considered over a period of five years, to determine whether it could be regarded as being due to a natural transport of inoculum from plant to plant. In order to do this it is essential to examine both spatial and temporal aspects of the spread.

The spatial approach is necessary for the consideration of whether diseased plants occur in agglomerations, or whether they can be regarded as occurring randomly in space. The hops in this instance were planted at a distance of 6 ft. square, and the method of training them was such that each plant touched its neighbours, along the rows, across the rows and diagonally. This meant that the queen case suggested itself as being the one to use. The result of the analysis was that there was strong evidence, in three out of the five years at least, that diseased plants were more often found in agglomerations than would have been expected had these plants been randomly distributed.

As an example of the use of the method Legg's data for the year 1951 will be considered. Fig. 1 shows the distribution of plants showing nettlehead symptoms (black plants), those without symptoms (white plants), and missing plants (vacancies). The lattice is 11×30 plants in area, and so $a = 41$, $b = 330$. Further, there are 122 plants with symptoms and 196 without symptoms, and so these are the respective values of n_1 and n_2.

The distribution of vacancies is such that $u = 11$, $v = 1$, $w = 0$, and there are no pairs of neighbouring vacancies, i.e. $p = 0$. We thus have

$$U = 11+1 = 12 \quad \text{and} \quad V = (8 \times 11)+(5 \times 1) = 93.$$

This gives $b - U = 318$, and, from equation (7),

$$A = (4 \times 330)-(3 \times 41)+2-93 = 1106.$$

We now have to consider the existent points with various numbers of neighbouring vacancies, and we find that $x_1 = 66$, $x_2 = 8$, $x_3 = 1$, $y_1 = 8$, while the remaining x's and y's, and all the z's, are zero. Thus, from equation (12),

$$W = (28 \times 11)+(10 \times 1)+[(7 \times 66)+(13 \times 8)+(18 \times 1)]+(4 \times 8) = 934,$$

and so, from equation (11)

$$B = 4[(7 \times 330)-(9 \times 41)+11]-934 = 6874.$$

Before the next stage of the computation it is convenient to find $A(A-1)-2B$, which equals $(1106 \times 1105)-6874 = 1208382$. For BB joins we then have

$$\mu = \frac{1106 \times 122 \times 121}{318 \times 317} = 162{\cdot}0$$

and $$\sigma^2 = \mu + \frac{2 \times 6874 \times 122 \times 121 \times 120}{318 \times 317 \times 316} + \frac{1208382 \times 122 \times 121 \times 120 \times 119}{318 \times 317 \times 316 \times 315} - \mu^2,$$

which reduces to 530·66, whence $\sigma = 23{\cdot}0$.

Similarly, for BW joins, we find $\mu = 524{\cdot}7$, $\sigma = 15{\cdot}8$.

The observed numbers of BB and BW joins are respectively 220 and 408, the former of which exceeds expectation by about two and a half times the standard error, and the latter of which falls short of expectation by more than seven times the standard error. By thus using the standard error as a measure of the discrepancy between theory and observation it appears from the combination of these two results that it is very unlikely that the observed distribution of diseased plants was in fact due to a random distribution of these plants in the field.

In the case of nettlehead, which is a virus disease probably transmitted by insects, and with this particular method of growing hops, the queen case of the spatial aspect must be

the one to use. This need not, however, always be the case, and there may be times when it is desirable to consider the spread of disease where diagonal joins are excluded. Furthermore, in the case of a disease believed to be spread by cultivations, these being carried out in one direction only, it may be appropriate to consider a linear lattice. (Since the formula for this case can be easily derived from that for a rectangular lattice by taking one of the dimensions

Fig. 1. Nettlehead status of plants in 1951

	39	40	41	42	43	44	45	46	47	48	49
1	.	.	.	.	.	N	V	N	.	N	N
2	.	N	.	.	N	N	N	.	N	.	.
3	N	N	N	N	.	N	.	N	N	N	.
4	N	N	.	N	N	N	N	N	N	.	.
5	N	N	N	N	.	N	.	.	N	N	.
6	.	N	N	.	N	N	N	.	.	N	N
7	N	N	N	N	.	N	.	.	.	.	.
8	N	.	V	N	.	N	N	N	N	N	N
9	N	N	.	N	V	.	N	N	.	N	.
10	.	N	N	N	N	N	N	N	V	.	N
11	N	N	N	.	.	.	N	N	.	.	N
12	.	.	N	.	.	N	.	N	N	N	N
13	N	N	V	N	.	N	.	.	N	.	N
14	N	.	N	N	.	N	V	.	V	.	.
15	.	V	.	.	.	.	.	.	.	N	.
16	.	.	.	.	N	.	V	.	N	N	N
17	.	.	.	.	.	.	.	.	.	V	N
18	.	.	N	.	N	.	.	.	.	N	.
19	N	.	.	N	.	.	.	.	.	.	.
20	.	.	.	N	.	.	.	.	N	.	.
21	.	.	V	.	.	N	.	.	N	.	.
22	.	.	.	.	.	N	.	N	N	.	N
23	.	.	.	.	N	V	.	.	.	.	N
24	.	.	N	.	.	.	.	.	.	.	N
25	.	.	.	.	.	.	.	N	.	N	N
26	.	.	.	.	.	.	.	.	N	N	N
27	.	.	.	.	.	.	.	.	.	N	N
28	N	.	.	.	.	.	.	.	.	N	.
29	.	.	.	.	.	.	.	.	.	N	.
30	.	.	.	.	.	.	.	.	.	.	.

N = Nettlehead plant. V = Vacancy. . = Plant without symptoms.

to be of unit width, it is not given explicitly here.) There may, moreover, be cases where practically nothing is known about the disease under investigation, and then it may be useful to consider a sort of sequential approach, first using a linear lattice, and then, successively, the rook and queen cases for the rectangular lattice. If this is done, and one method shows evidence of spread but the next does not, the results may well shed some light on the method of spread of the disease in question, and thus provide clues to the agency of spread.

Investigation of the spatial aspect of the matter is not in general enough by itself, and the temporal approach should be used in conjunction with it. The reason for this is that even definite evidence from the spatial approach cannot do more than suggest that diseased plants occur in clusters. This is not sufficient to prove that there is infectious spread, for there may be ecological or other factors which could give rise to the same pattern of diseased plants. For example, there could be localized conditions of soil or shelter giving rise to patches where the disease develops or where symptoms are expressed. Equally a

physiological disorder can occur in patches. If, however, the temporal method shows that the disease is spreading throughout the plantation under investigation, while retaining the characteristic pattern of agglomerations, there would be strong evidence of genuinely infectious spread.

The investigation of nettlehead disease provides one practical instance of the use of the two approaches, and a second example is given by another disease of hops, the fluctuating type of *Verticillium* wilt. In one particular locality the appearance in one year of agglomerations of diseased plants suggested a spread of the disease, but these agglomerations were found to be due to the influence of flood water in the winter in relation to the contour pattern in this hop garden. Here the use of the temporal approach would have immediately shown the spread to be apparent rather than real.

The temporal method is used to examine whether plants next to diseased plants are more liable to go down with a disease than those all of whose neighbours are healthy. To revert to the nettlehead example, the investigation concerned itself with the virus status in any year of plants healthy in the preceding year, in order to see if the disease spread principally to those plants whose neighbours were already diseased. Any plant which was missing the second year but not diseased the first year was tacitly excluded.

To investigate this, the method of Barnard's 2×2 comparative trial was used. For, as Barnard states, the fundamental subject-matter of a 2×2 comparative trial is two populations, and here we have the two populations of healthy plants: (i) those having some diseased neighbours, and (ii) those having all healthy neighbours. These populations being given, records are then taken in the next year to see whether the proportion of diseased plants is the same in the two populations. It will be seen that, whereas there may be some doubt as to which type of spatial approach to use, the essential supplement using the temporal approach must make use of the 2×2 comparative trial. It is obvious that the same definition of 'neighbours' will obtain for the two aspects of the same investigation.

Fig. 2 shows the distribution of plants showing nettlehead in 1951 and not in 1950 in relation to plants that showed nettlehead in 1950. The division of the population described above shows the following numbers of plants in each category:

Neighbours in 1950	Plants in 1951 healthy in 1950		
	Diseased	Healthy	Total
Some diseased	53	88	141
All healthy	12	90	102
Total	65	178	243

In Barnard's notation, u being a variate distributed normally with unit variance, we then have

$$u = \frac{(53 \times 90) - (12 \times 88)}{243} \div \sqrt{\frac{141 \times 102 \times 65 \times 178}{243 \times 243 \times 242}},$$

whence $u = 4{\cdot}48$ with probability $P < 0{\cdot}001$.

It thus appears that diseased plants are very much more likely to occur if some of their neighbours were previously diseased than if all their neighbours were previously healthy, which suggests that there is some form of infectious spread taking place.

In the nettlehead example, the results of the investigation of the temporal aspect agreed very closely with those of the spatial aspect because those years when the spatial approach

Fig. 2. Nettlehead status of plants in 1951 in relation to that of 1950

	39	40	41	42	43	44	45	46	47	48	49
1	.	.	.	.	.	O	V	O	O	O	O
2	.	N	.	.	N	N	O	.	O	.	.
3	O	O	N	N	O	O	.	N	O	N	.
4	O	O	.	N	O	O	O	N	N	O	.
5	N	N	N	O	O	N	O	.	O	O	O
6	.	O	O	.	N	O	N	.	O	O	O
7	O	N	O	O	.	O	.	.	.	.	O
8	N	.	V	N	O	O	N	O	N	N	N
9	N	O	O	O	V	O	N	O	O	N	.
10	.	N	N	O	O	N	N	O	O	.	O
11	O	N	O	O	.	.	O	N	.	.	N
12	.	.	N	.	.	O	.	N	N	O	N
13	O	N	V	O	.	N	.	.	N	O	N
14	O	.	N	N	.	N	V	.	V	.	.
15	.	V	O	.	O	.	.	.	.	O	.
16	.	.	.	.	O	.	V	.	O	O	O
17	.	.	.	.	O	.	.	.	.	V	N
18	.	.	N	.	N	.	.	.	.	O	.
19	N	.	.	N	.	.	.	.	.	.	.
20	.	.	.	N	.	.	.	.	N	.	.
21	.	.	V	.	.	N	.	.	N	.	.
22	.	.	.	.	.	N	.	N	N	.	N
23	.	.	.	.	O	V	.	.	.	.	N
24	.	.	O	.	.	.	.	.	.	.	O
25	.	.	.	.	.	.	.	N	.	N	O
26	.	.	.	.	O	.	.	.	N	O	O
27	.	.	.	.	.	.	.	.	.	N	N
28	N	.	.	.	.	.	.	.	.	N	.
29	.	.	.	.	.	.	.	.	.	N	.
30	.	.	.	.	.	.	.	.	.	.	.

O = Nettlehead plant in 1950, irrespective of whether it had nettlehead, had been replanted or was a vacancy in 1951.
N = Nettlehead plant in 1951 but not 1950.
V = Vacancy in 1951, not nettlehead in 1950.
. = Plant without symptoms in both 1951 and 1950.

suggested agglomeration were those in which there was a tendency for more newly diseased plants to be next to those diseased the previous year. Also, in this instance, the disease had spread so far across the recorded area by the end of the period under investigation as to suggest that the liability of plants to infection was not related to their position, and thus there was less likelihood of ecological factors accounting for the spread. Further, in general, when there were significantly more *BB* joins than expectation there were significantly fewer *BW* joins, and when one did not differ significantly from expectation neither did the other.

In most practical cases plants will be either diseased or healthy, and so the black and white colours are likely to account for all the existent points on the lattice. There could,

however, be cases, such as those of a complex of viruses, where more colours were possible. Also, there could be cases where some of the plants were carriers of the disease without showing symptoms themselves. It would then probably be desirable to treat such plants as vacancies from the spatial point of view but as diseased neighbours the previous year in the temporal approach.

4. Summary

This paper considers the spread of disease in a rectangular plantation where there may be some missing plants, or vacancies. The problem is considered from two points of view, spatial and temporal. In the former, formulae are given which modify the existing methods derived by Krishna Iyer to allow for vacancies. In the latter, the 2×2 comparative trial of Barnard is used. Both methods are applied to a particular example of the apparent spread of a hop virus disease, nettlehead. In this case the results achieved by the two methods supplement each other, and, in general, both must be used before it can be concluded if the apparent spread corresponds to a spread of inoculum.

I am much indebted to Dr S. C. Pearce for helpful discussion on the statistical aspects, to Mr J. T. Legg for his ready co-operation on the plant pathology side of the study and to Dr R. V. Harris for bringing to my notice the example relating to *Verticillium* wilt.

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TESTS OF SIGNIFICANCE FOR CONCURRENT REGRESSION LINES

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The fitting of a straight regression line to a set of observations on two variates is a straightforward application of the method of least squares. Where there are several sets of such data, it is of interest to fit regression lines for each set, and to determine whether or not the slopes of the lines differ more than would be expected due to chance variations. Where the slopes of a set of lines do not differ significantly, the interpretation of the data is particularly straightforward, since the comparison of the sets may be made simply by comparing the constant terms of the regression equations. In fact, if these terms also do not differ significantly when a single slope is used for all lines, then the regression may be regarded as completely homogeneous, and a single equation used to represent the relationship for all the data.

The case of parallel regression lines is particularly important in many applications; for example, the comparison of two materials by means of biological assay can be carried out satisfactorily only when the regression of response (suitably measured) on dosage is linear, and has the same slope for each material. The relative potency is then measured as the distance between the parallel lines, measured in a direction parallel to the dosage axis. When the regression lines are not parallel the interpretation is difficult or impossible.

Another case which appears to be important but has not received so much attention is the case in which the regression lines, instead of being parallel, are concurrent. This type of effect is likely to occur when different materials produce different rates of response, while all give the same response at some fixed level. This level in practice is often not zero, and often not exactly known. When three or more regression lines are concurrent, it is possible to measure the difference between the effects of different materials or treatments by means of the ratios of the distances between the lines measured along an ordinate. It will be seen that parallel regression lines are simply a special case of concurrent lines; in fact, the ratio of the effectiveness of different materials may be measured in precisely the same way as for concurrent lines, that is, by the distances between the intercepts on a transversal.

Tocher (1952) has recently described situations in which it is of interest to know whether the regression lines for different sets of experimental results are concurrent. He has given a test of significance of the hypothesis of concurrence, and a method of determining fiducial limits for the abscissa (i.e. the value of the independent variable) of the point of concurrence. The purpose of this note is to give what is considered to be a more satisfactory solution of these two problems, as well as to discuss further applications of the fitting of concurrent lines.

When the point of concurrence of the lines is known in advance, or only the ordinate requires estimation, the problem is, as pointed out by Tocher, amenable to standard methods. When the abscissa requires estimation, the sums of squares required for the significance tests given here are found to be latent roots of a certain matrix. The analysis and tests of significance are found to parallel those given (Williams, 1952*b*) for the interpretation of interactions in factorial experiments.

FORMULATION OF THE PROBLEM

Our notation differs from that of Tocher, which is not always convenient for the present discussion. We consider m sets of values of x and y; the pairs of values, n_i in number, in the ith set are denoted by x_i and y_i. We confine ourselves to the special case

$$\left.\begin{aligned}\Sigma x_i &= \bar{x} n_i,\\ \Sigma x_i^2 &= X^2 n_i,\end{aligned}\right\} \tag{1}$$

($\bar{x}$ and X^2 being constants; $X \neq 0$ since the x_i cannot all vanish), which includes the commonly occurring case of all sets of data having the same values of x. Without loss of generality, in the mathematical discussion we take

$$\bar{x} = 0, \tag{2}$$

$$\sum_i \Sigma y_i = 0. \tag{3}$$

We also put

$$\Sigma y_i = n_i \bar{y}_i, \tag{4}$$

$$\Sigma y_i x_i = n_i q_i, \tag{5}$$

$$\sum_i n_i = N, \tag{6}$$

$$\sum_i n_i q_i = Nq, \tag{7}$$

and take the point of concurrence as (ξ, η).

(i) *Ordinate of concurrence given*

We assume that η is known, and only ξ is to be estimated. The regression coefficient for the ith line through (ξ, η) is

$$b_i = \frac{\Sigma(y_i - \eta)(x_i - \xi)}{\Sigma(x_i - \xi)^2}$$

$$= \frac{q_i - \xi(\bar{y}_i - \eta)}{X^2 + \xi^2}. \tag{8}$$

The sum of squares for regressions through (ξ, η), with m degrees of freedom, is accordingly

$$S = \sum_i n_i b_i^2 (X^2 + \xi^2)$$

$$= \frac{\sum_i n_i q_i^2 - 2\xi \sum_i n_i q_i (\bar{y}_i - \eta) + \xi^2 \sum_i n_i (\bar{y}_i - \eta)^2}{X^2 + \xi^2}. \tag{9}$$

Since the total sum of squares about η is independent of ξ, to estimate ξ we may equally well maximize S as minimize the residual sum of squares. Put

$$\frac{\Sigma n_i q_i^2}{X^2} = A, \quad \frac{\Sigma n_i q_i (\bar{y}_i - \eta)}{X} = B, \quad \Sigma n_i (\bar{y}_i - \eta)^2 = C.$$

Then (9) gives

$$(X^2 + \xi^2) S = AX^2 - 2BX\xi + C\xi^2, \tag{9'}$$

and S is maximized if

$$\xi S = -BX + C\xi \tag{10}$$

or

$$XS = AX - B\xi. \tag{10'}$$

From (10) and (10') we see that the maximized S must be the larger latent root of the matrix

$$\begin{bmatrix} A & -B \\ -B & C \end{bmatrix}. \tag{11}$$

The two latent roots, S_1 and S_2, may be given a practical interpretation. S_1 is the sum of squares for regressions and the point of concurrence, and thus has $m+1$ degrees of freedom. S_2, with $m-1$ degrees of freedom, is the sum of squares representing departures from concurrence. Corresponding to the latent root S_1, we have, from (10),

$$\xi = -\frac{\sum_i n_i q_i(\bar{y}_i - \eta)}{S_1 - \sum_i n_i(\bar{y}_i - \eta)^2}$$

$$= -\frac{XB}{S_1 - C}. \tag{12}$$

From the form of the matrix we see that S_2 is zero if and only if

$$q_i \propto \bar{y}_i - \eta, \tag{13}$$

and from (12), that in such cases the coefficient of proportionality is $-X^2/\xi$. Then $b_i = q_i/X^2$, from (8), and the unrestricted regression lines for each set all pass through the point of concurrence. In particular, there is no departure from concurrence when either all the q_i are zero, so that the lines are parallel, or all the $\bar{y}_i = \eta$, so that the point of means is the point of concurrence.

(ii) *Ordinate of concurrence to be estimated*

When η is not given, it may be so chosen as to minimize the residual sum of squares. We then find

$$\eta = \xi q/X^2. \tag{14}$$

But q/X^2 is the average unrestricted regression coefficient for all the data; therefore, as might be expected, the point of concurrence lies on the line of mean regression when η is estimated from the data. It is therefore appropriate to consider departures from mean regression only in testing the significance of departures from concurrence.

The sum of squares for departures of individual regressions from mean regression is

$$S' = (\sum_i n_i b_i^2 - Nb^2)(X^2 + \xi^2)$$

$$= \frac{\sum_i n_i(q_i - q)^2 - 2\xi \sum_i n_i \bar{y}_i(q_i - q) + \xi^2 \sum_i n_i \bar{y}_i^2}{X^2 + \xi^2}$$

$$= \frac{A'X^2 - 2B'X\xi + C'\xi^2}{X^2 + \xi^2} \quad \text{(say)}, \tag{15}$$

with $m-1$ degrees of freedom. As with S, the maximum value of S' is found as the larger latent root, with m degrees of freedom, of

$$\begin{bmatrix} A' & -B' \\ -B' & C' \end{bmatrix}. \tag{16}$$

Again, the smaller latent root, with $m-2$ degrees of freedom, provides a test of departures from concurrence. The effect of having to estimate η is merely to replace the individual means and regressions by departures from overall means and regressions, and to reduce by one the degrees of freedom.

Tests of significance

In testing the significance of the differences among regression lines for several sets of data, the usual approach is to examine, first, whether the slopes of the lines differ significantly, and secondly, whether, if the slopes are assumed not to differ, the distances between the parallel lines fitted to the data differ significantly. For the present purpose, the two aspects are viewed in a different way. It is seen from the above discussion that the sum of the two latent roots involves both the differences among regressions and the differences among means, but that the larger latent root, representing differences among regression lines, may involve one or other aspect alone.

The testing of significance of the latent roots of a matrix has been discussed elsewhere (see, for example, Williams, 1952*a*, *b*, and references). The kind of test which seems appropriate to the present problem is one which, first, tests the validity of the assumed model, i.e. whether or not the departure from concurrence is significant, and secondly, gives fiducial limits for ξ. We first consider case (i) above, with η known. Corresponding to a given value of ξ we may determine S, the sum of squares for regressions through (ξ, η), then calculate the quantities

$$v_1 = \frac{(S_1 - S)(S - S_2)}{S} \tag{17}$$

and

$$v_2 = \frac{S_1 S_2}{S}. \tag{18}$$

These are distributed as sums of squares with 1 and $m-1$ degrees of freedom respectively. v_2 gives an exact test of departures from concurrence, corresponding to ξ, and v_1 a test of concordance of the point of concurrence (ξ, η) with the data. Provided v_2 is not significant, v_1 may be used to give a fiducial limit for S at any given probability level, from (17).

It should be emphasized that, although it is often still possible formally to determine fiducial limits for ξ even when v_2 is significant, such fiducial limits for the abscissa of concurrence are irrelevant, since the hypothesis of concurrence is no longer tenable. It is therefore important to test v_2 first before considering the estimation of ξ.

The corresponding fiducial limits for ξ are derived from equations (9) and (17), and are most conveniently given as follows:

$$\xi^2[S - \sum_i n_i(\bar{y}_i - \eta)^2] + 2\xi \sum_i n_i q_i(\bar{y}_i - \eta) + X^2 S - \sum_i n_i q_i^2 = 0,$$

$$\xi = -\frac{\sum_i n_i q_i(\bar{y}_i - \eta) \pm X\sqrt{(S v_1)}}{S - \sum_i n_i(\bar{y}_i - \eta)^2}$$

$$= -\frac{X(B \pm \sqrt{(S v_1)})}{S - C}. \tag{19}$$

Alternatively, since v_1 is a sum of squares with 1 degree of freedom, its numerator may be expressed as the square of a rational function of ξ. Since, from (12), v_1 vanishes when

$$\xi = -\frac{XB}{S_1 - C}$$

or

$$\xi = \frac{XB}{C - S_2},$$

it appears that
$$Sv_1 = \frac{[(S_1-C)\xi+XB]^2[(C-S_2)\xi-XB]^2}{B(X^2+\xi^2)^2}$$
$$= \frac{[\xi^2B+\xi X(C-A)-X^2B]^2}{(X^2+\xi^2)^2}. \tag{20}$$

The non-linear analysis of variance may be set out as follows:

	Degrees of freedom	Sums of squares
Point of concurrence	1	$\frac{(S_1-S)(S-S_2)}{S}$
Regressions through point of concurrence	m	S
Departures from concurrence	$m-1$	$\frac{S_1S_2}{S}$
Total between lines	$2m$	S_1+S_2
Residual	$N-2m$	by difference
Total	N	$\sum_i \Sigma (y_i-\eta)^2$

When $m = 1$, $S_2 = 0$, so that the test for a given point of concurrence reduces to a test of significance of the difference $S_1 - S$.

For case (ii), when η is to be estimated, a similar approach may be used. Here, however, we require to determine *simultaneous* fiducial limits for ξ and η. The limits will be given as a closed curve surrounding the optimum point.

If the variation of η about the value given by (14) is not considered, we may calculate the statistics
$$v_1' = \frac{(S_1'-S')(S'-S_2')}{S'} \tag{21}$$
and
$$v_2' = \frac{S_1'S_2'}{S'}, \tag{22}$$
with 1 and $m-2$ degrees of freedom respectively. As for the case when η is known, these two statistics may be used to test departures from the abscissa of concurrence ξ, and departures from concurrence, respectively.

Now for the departure of a given value of η from its optimum value $\xi q/X^2$, the sum of squares, with 1 degree of freedom, is
$$\frac{N(\eta X^2-\xi q)^2}{X^2(X^2+\xi^2)}. \tag{23}$$

For a simultaneous test of significance of the concordance of ξ and η with the data we may therefore take
$$v_1' + \frac{N(\eta X^2-\xi q)^2}{X^2(X^2+\xi^2)}$$
$$= S_1'+S_2'-S'-\frac{S_1'S_2'}{S'}+\frac{N(\eta X^2-\xi q)^2}{X^2(X^2+\xi^2)}$$
$$= S_1+S_2-S-\frac{S_1'S_2'}{S'}$$
$$= v_1+v_2-v_2'. \tag{24}$$

This is a sum of squares with 2 degrees of freedom. The set of values of ξ and η giving a constant value to (24) are the required fiducial limits for the position of the point of concurrence.

Comparison with Tocher's results

The present paper gives significance tests which are exact, and more detailed than those given by Tocher. In fairness it should be remarked, however, that we have treated only the special case defined by (1), and that there are unlikely to be exact tests for the general case treated by Tocher. As there are similarities between the two approaches, it is of interest to compare them. For convenience, we rewrite some of Tocher's results in our notation; his equation numbers will be distinguished by the suffix T.

Tocher first tests the significance of the restraint that the lines be concurrent at a point with abscissa ξ. The criterion is a sum of squares with m degrees of freedom, which is identical with our

$$S_1+S_2-S = v_1+v_2. \tag{25}$$

Its minimum value, which Tocher takes as critical for the test (14T), is clearly S_2, which is reached when $S = S_1$, and ξ has its optimum value. S_2, however, is not distributed as a sum of squares with m degrees of freedom, as Tocher assumes; nor would a reduction of degrees of freedom to $m-1$ be correct, since the equation for optimum ξ is not linear. Tocher's test is, in fact, conservative, as he intends; that is, the significance of departures from concurrence tends to be underestimated (see Williams, 1952*a*, p. 23).

The optimum value of ξ is that for which $S = S_1$. For fiducial limits, Tocher uses the values of ξ corresponding to a value of S making v_1+v_2 significant at a given probability level (15T). The values of ξ are in fact given by (19), with the appropriate value of S. We propose that the setting of limits be done in two stages. First, corresponding to any ξ, v_2, representing departures from concurrence, is tested for significance. If v_2 is not significant, fiducial limits for ξ are determined from the value of v_1 significant at a given probability level. If v_2 is significant, the hypothesis of concurrence must be rejected, and no meaning can be attached to a point of concurrence.

Either method of setting fiducial limits is theoretically correct. However, it is considered that there is a conceptual advantage in distinguishing between the test of the hypothesis of concurrence and the test of a given point of concurrence. When v_2 is small, the use of v_1 also gives narrower fiducial limits than v_1+v_2. This point is illustrated in the numerical example given below.

Numerical examples

(i) *Re-examination of Tocher's data*

In order to compare the results given above with Tocher's, we now carry out a comparative analysis of his results. It is noted that he takes $\eta = 0$. From his data (p. 115) we find

$$\begin{aligned} A &= 0{\cdot}1013\,5806\,6, \\ B &= 0{\cdot}9224\,9221\,3, \\ C &= 8{\cdot}3960\,7140\,0, \\ S_1+S_2 &= 8{\cdot}4974\,2946\,6, \\ S_1S_2 &= 0{\cdot}0000\,1767\,7, \\ S_1 &= 8{\cdot}4974\,2738\,6, \\ S_2 &= 0{\cdot}0000\,0208\,0. \end{aligned}$$

Residual mean square $= 0{\cdot}9223\,1 \times 10^{-6}$ (6 d.f.).

Since the 5 and 1 % points of F, with 1 and 6 degrees of freedom, are 5·9874 and 13·745 respectively, a sum of squares with 1 degree of freedom, to be significant, must exceed

$$5{\cdot}5222 \times 10^{-6} \quad (5\,\% \text{ level}),$$

or

$$12{\cdot}677 \times 10^{-6} \quad (1\,\% \text{ level}).$$

Since, for all values of S considered henceforth, the value of v_2 is not significant, departures from concurrence will be assumed non-existent.

We now determine fiducial limits for S, and hence for ξ, from the equation

$$v_1 = 8{\cdot}4974\,2946\,6 - S - \frac{0{\cdot}0000\,1767\,7}{S} = 5{\cdot}5222 \times 10^{-6}$$

and

$$= 12{\cdot}677 \times 10^{-6}.$$

The fiducial limits are as follows:

	Present method		Tocher's method	
	S	ξ	S	ξ
Optimum value	8·4974 27386	981·5	8·4974 27386	981·5
5 %	21864	970 and 993	17899	968 and 995
1 %	14709	964 and 999	07234	961 and 1002

The limits by Tocher's method, given for comparison, are found from (15T), or from equation (19) with the appropriate value of S. These ranges are 17 % wider than those from the present method, owing to the inclusion of v_2 in the determination of S.

(ii) *Data requiring estimated ordinate*

In a study of the effect of basis weight on the properties of laboratory-made sheets of paper, three batches of pulp were taken and beaten for different lengths of time. From each batch, sheets of six different basis weights were then made, and mechanical tests carried out on them. The burst strength results from this experiment are set out in Table 1, together with values calculated from them.

Table 1. *Burst strength results*

Basis weight	x	Beating (revs of Lampen Mill)			Total
		1125 $i = 1$	4500 $i = 2$	12730 $i = 3$	
10	−5	1·73	1·98	3·11	
20	−3	4·99	9·69	12·83	
30	−1	8·74	17·26	22·69	
40	1	12·60	24·52	32·45	
50	3	17·04	31·36	48·22	
60	5	21·18	43·29	59·94	
$6\bar{y}_i$		66·28	128·10	179·24	373·62
$6q_i$		137·26	278·82	400·08	816·16

Table 2. *Test of departures of regression lines from origin*

	Degrees of freedom	Sums of squares	Mean square
Correction for mean	3	8821·604	
Regression through means	3	3666·356	
Sum	6	12487·960	
Regressions through origin	3	12311·932	
Departures from origin	3	176·028	58·676**
Residual	12	28·554	2·3795

** Significant at 1 % level.

A preliminary analysis, as given in Table 2, shows that the three regression lines of burst strength on basis weight depart significantly from the origin. In order to compare the trends for different amounts of beating it would be desirable to base the comparison on regression lines through some other common point, if such a model were concordant with the data. The common point would represent the basis weight for which the value of burst factor was independent of beating. The ordinate as well as the abscissa of this point will be estimated.

For the independent variable we take

$$x = \frac{\text{basis weight} - 35}{5},$$

so that x has the values $-5, -3, -1, 1, 3, 5$, and $X^2 = \frac{35}{3}$. The quantities required for the analysis are given in Table 1.

The mean burst strength is 20·76, and the mean regression coefficient on x is

$$b = \frac{816{\cdot}16}{210} = 3{\cdot}886,$$

so that, by (14), the optimum value of the ordinate corresponding to ξ is

$$\eta = 3{\cdot}886\xi + 20{\cdot}76.$$

The matrix (16) is

$$\begin{bmatrix} 494{\cdot}36940 & -726{\cdot}07978 \\ -726{\cdot}07978 & 1066{\cdot}49853 \end{bmatrix},$$

whence

$$S_1' = 1560{\cdot}83437,$$

$$S_2' = \quad 0{\cdot}03356.$$

The optimum value of (ξ, η) is $(-5{\cdot}02, 1{\cdot}26)$ from (12) and (14). Departures from concurrence are clearly not significant, so attention may be confined to $v_1 + v_2 - v_2'$ and the determination of the fiducial boundary for (ξ, η). From the residual mean square given in Table 2, the values for $v_1 + v_2 - v_2'$ required for significance may be determined as

$$18{\cdot}490 \quad (5\,\% \text{ level}),$$

and

$$32{\cdot}964 \quad (1\,\% \text{ level}).$$

The equation for the fiducial boundary is, then,

$$\frac{(726{\cdot}1\xi^2+1954\xi-8471)^2}{(\xi^2+\frac{35}{3})(1066\xi^2-4960\xi+5768)} + \frac{210(\eta-3{\cdot}886\xi-20{\cdot}76)^2}{(\xi^2+\frac{35}{3})} = 18{\cdot}490 \quad \text{and} \quad 32{\cdot}964.$$

Of particular interest are the maximum ranges for ξ and η. The maximum range for ξ is given above when η does not depart from the value given by the mean regression line. Then, from (19),

$$S = 1542{\cdot}34, \quad \xi = -6{\cdot}4 \text{ and } -4{\cdot}0 \quad (5\,\% \text{ level}),$$
$$S = 1527{\cdot}87, \quad \xi = -7{\cdot}0 \text{ and } -3{\cdot}7 \quad (1\,\% \text{ level}).$$

The maximum range for η is given when ξ has its optimum value $-5{\cdot}02$; negative values being inadmissible, the limits are found to be

$$0 \text{ and } 3{\cdot}1 \quad (5\,\% \text{ level}),$$
$$0 \text{ and } 3{\cdot}7 \quad (1\,\% \text{ level}).$$

CONCLUSION

The main emphasis of this paper has been to show that the concepts developed for tests of significance in multivariate analysis and for interactions may be carried over to dealing with a problem of concurrent regressions. It would clearly be possible to extend this work to regression analysis with two or more independent variates. The quantities required for testing significance would then, for k variates, come out as latent roots of a $(k+1)\times(k+1)$ matrix. Such a generalization, however, appears to lack practical interest.

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APPROXIMATE CONFIDENCE INTERVALS

II. MORE THAN ONE UNKNOWN PARAMETER

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1. Retrospect

In a previous paper (Bartlett, 1953; this will be referred to as I) it was shown that the approximate confidence interval obtained from the asymptotic normal distribution for $\partial L/\partial\theta$, where $L \equiv \log p$ is the log likelihood depending on a single unknown parameter θ, could usually be improved by the use of skewness or further moment corrections. The purpose of the present paper is to examine the further problems that arise in the case of more than one unknown parameter. It is perhaps advisable to point out that while the direct use of quantities like L and its derivatives permits a fairly flexible sampling basis, the asymptotic normal distribution remains the basis of the approximations. We may note (see § 7) the application of the proposed method to non-classical problems such as time-series analysis. It could not, however, be used in problems for which the central limit theorem did not apply. This remark appears relevant if we consider (see §§ 5 and 6) some of the more controversial problems involving nuisance parameters, as R. A. Fisher's fiducial probability solutions have been given by some writers (see, for example, Barnard, 1950) an approximate frequency interpretation by means of special (and in my opinion rather artificial*) sampling interpretations.

2. Simultaneous moments

The systematic method of obtaining relations between the moments of $\partial L/\partial\theta$, $\partial^2 L/\partial\theta^2$, etc., is obviously extensible to the case of several unknowns $\theta_i\,(i = 1, 2, \ldots)$. The derivation of the more general formulae rapidly becomes rather tedious, and I shall only record the formulae up to the third-order moments of $\partial L/\partial\theta_i$. On the usual assumptions about differentiation with respect to θ_i commuting with the integral sign, we may write (cf. I)

$$1 = \int p(\theta_i + \tau_i) = \int p(\theta_i) \exp\left\{\Sigma_i \tau_i \frac{\partial L}{\partial \theta_i} + \tfrac{1}{2}\Sigma_i \Sigma_j \tau_i \tau_j \frac{\partial^2 L}{\partial\theta_i \partial\theta_j} \cdots\right\},$$

whence

$$E\left\{\frac{\partial L}{\partial \theta_i}\right\} = 0, \quad E\left\{\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}\right\} = E\left\{-\frac{\partial^2 L}{\partial\theta_i\partial\theta_j}\right\},$$

$$E\left\{\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}\frac{\partial L}{\partial\theta_k}\right\} = E\left\{-\frac{\partial^3 L}{\partial\theta_i\partial\theta_j\partial\theta_k}\right\} - E\left\{\frac{\partial L}{\partial\theta_i}\frac{\partial^2 L}{\partial\theta_j\partial\theta_k}\right\} - E\left\{\frac{\partial L}{\partial\theta_j}\frac{\partial^2 L}{\partial\theta_i\partial\theta_k}\right\} - E\left\{\frac{\partial L}{\partial\theta_k}\frac{\partial^2 L}{\partial\theta_i\partial\theta_j}\right\}.$$

Differentiating the second-order equation, we see further that

$$\frac{\partial}{\partial\theta_i} E\left\{\frac{\partial^2 L}{\partial\theta_j\partial\theta_k}\right\} + E\left\{\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}\frac{\partial L}{\partial\theta_k}\right\} + E\left\{\frac{\partial^2 L}{\partial\theta_i\partial\theta_j}\frac{\partial L}{\partial\theta_k}\right\} + E\left\{\frac{\partial^2 L}{\partial\theta_i\partial\theta_k}\frac{\partial L}{\partial\theta_j}\right\} = 0.$$

* It is not my purpose to cloud the present discussion with any controversial fog. I would, however, note that the sampling procedure suggested by Barnard in the Behrens-Fisher problem, if deliberately carried out, would lead approximately to the solution claimed, but at the cost of throwing away the information on the variances available in the expected further observations. It seems to me that we should rarely want to do this unless the resulting loss of accuracy in our inference on the mean difference was negligible, but this is just the limiting situation for which we do not require small-sample theory at all.

From these last two equations, we find that

$$E\left\{\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}\frac{\partial L}{\partial\theta_k}\right\} = 2E\left\{\frac{\partial^3 L}{\partial\theta_i\partial\theta_j\partial\theta_k}\right\} + \frac{\partial}{\partial\theta_i}I_{jk} + \frac{\partial}{\partial\theta_j}I_{ik} + \frac{\partial}{\partial\theta_k}I_{ij}, \tag{1}$$

where $I_{ij} \equiv E\{-\partial^2 L/\partial\theta_i\partial\theta_j\}$. Formulae of the same order involving less than three parameters may be obtained from (1) by contraction (i.e. making two or more θ's identical).

3. CONFIDENCE REGIONS

As already noted in I, the extension from approximate confidence intervals to confidence regions is in principle straightforward. Thus for two parameters θ_1 and θ_2 our first large-sample approximation is to consider the joint quantities $\partial L/\partial\theta_1$, $\partial L/\partial\theta_2$ as normal with zero means and known covariance-matrix. From these we may, for example, construct an approximate χ^2 expression, writing this (for convenience later) in the unsymmetric form

$$\left(\frac{\partial L}{\partial\theta_1}\right)^2 \Big/ I_{11} + \left(\frac{\partial L}{\partial\theta_2} - \frac{I_{12}}{I_{11}}\frac{\partial L}{\partial\theta_1}\right)^2 \Big/ \left(I_{22} - \frac{I_{12}^2}{I_{11}}\right). \tag{2}$$

Values of θ_1 and θ_2 for which the expression in (2) remains less than χ_0^2, the critical value of χ^2 for two degrees of freedom, define an approximate confidence region. An even simpler expression asymptotically equivalent to (2) is $-2[L - L_{\max}]$, discussed more fully later.*

It is, perhaps, hardly necessary to consider any skewness corrections to (2), for in it only *squares* of the approximately normal quantities are involved. Any skewness corrections would merely help to locate the minimum and maximum 'admissible' values of θ_1 and θ_2 more symmetrically (in terms of probability contributions to the total probability of error). However, to illustrate the procedure these corrections will be deduced. Consider

$$\left.\begin{aligned} T_1(\theta_1,\theta_2) &\equiv \frac{\partial L}{\partial\theta_1} + \lambda_1\left[\left(\frac{\partial L}{\partial\theta_1}\right)^2 - I_{11}\right], \\ T_2(\theta_1,\theta_2) &\equiv \frac{\partial L}{\partial\theta_2} - \frac{I_{12}}{I_{11}}\frac{\partial L}{\partial\theta_1} + \lambda_{20}\left[\left(\frac{\partial L}{\partial\theta_1}\right)^2 - I_{11}\right] + \lambda_{11}\left[\frac{\partial L}{\partial\theta_1}\frac{\partial L}{\partial\theta_2} - I_{12}\right] + \lambda_{02}\left[\left(\frac{\partial L}{\partial\theta_2}\right)^2 - I_{22}\right]. \end{aligned}\right\} \tag{3}$$

In order to replace the random variables in (2) by more nearly normal variables, we require the third-order cumulants of T_1 and T_2 to be zero to the next-order approximation. This gives

$$\left.\begin{aligned} &\kappa_{30} + 6\lambda_1 I_{11}^2 = 0 \quad \text{(cf. I)}, \\ &\left(\kappa_{21} - \frac{I_{12}}{I_{11}}\kappa_{30}\right) + 2\lambda_{20}I_{11}^2 + 2\lambda_{11}I_{11}I_{12} + 2\lambda_{02}I_{12}^2 = 0, \\ &\left(\kappa_{12} - 2\frac{I_{12}}{I_{11}}\kappa_{21} + \frac{I_{12}^2}{I_{11}^2}\kappa_{30}\right) + 2\lambda_{11}(I_{11}I_{22} - I_{12}^2) + 4\lambda_{02}I_{12}\left(I_{22} - \frac{I_{12}^2}{I_{11}}\right) = 0, \\ &\left(\kappa_{03} - 3\frac{I_{12}}{I_{11}}\kappa_{12} + 3\frac{I_{12}^2}{I_{11}^2}\kappa_{21} - \frac{I_{12}^3}{I_{11}^3}\kappa_{30}\right) + 6\lambda_{02}\left(I_{22} - \frac{I_{12}^2}{I_{11}}\right)^2 = 0. \end{aligned}\right\} \tag{4}$$

The last three equations may be solved successively from the bottom to the top equation; in the particular case when $I_{12} = 0$ they become

$$\kappa_{21} + 2\lambda_{20}I_{11}^2 = 0, \quad \kappa_{12} + 2\lambda_{11}I_{11}I_{22} = 0, \quad \kappa_{03} + 6\lambda_{02}I_{22}^2 = 0. \tag{5}$$

* It is convenient to defer discussion of this alternative until §8, as so far no explicit use of maximum-likelihood estimates has been necessary, although we shall find their introduction in general unavoidable from §4 onwards.

Thus for a normal sample with

$$L = -\frac{n}{2}\log\theta_2 - \frac{n}{2}\frac{(\bar{x}-\theta_1)^2}{\theta_2} - \frac{1}{2}\frac{(n-1)s^2}{\theta_2}, \tag{6}$$

so that

$$\frac{\partial L}{\partial\theta_1} = \frac{n(\bar{x}-\theta_1)}{\theta_2}, \quad \frac{\partial L}{\partial\theta_2} = -\frac{n}{2\theta_2} + \frac{(n-1)s^2 + n(\bar{x}-\theta_1)^2}{2\theta_2^2},$$

$$-\frac{\partial^2 L}{\partial\theta_1^2} = \frac{n}{\theta_2}, \quad -\frac{\partial^2 L}{\partial\theta_1\partial\theta_2} = \frac{n(\bar{x}-\theta_1)}{\theta_2^2}, \quad -\frac{\partial^2 L}{\partial\theta_2^2} = -\frac{n}{2\theta_2^2} + \frac{(n-1)s^2 + n(\bar{x}-\theta_1)^2}{\theta_2^3},$$

$$\left.\begin{aligned} &I_{11} = \frac{n}{\theta_2}, \quad I_{12} = 0, \quad I_{22} = \frac{n}{2\theta_2^2}, \\ &\kappa_{30} = 0, \quad \kappa_{21} = \frac{n}{\theta_2^2}, \quad \kappa_{12} = 0, \quad \kappa_{03} = \frac{n}{\theta_2^3}, \end{aligned}\right\} \tag{7}$$

we put

$$\lambda_1 = \lambda_{11} = 0, \quad \lambda_{20} = -\frac{1}{2n}, \quad \lambda_{02} = -\frac{2\theta_2}{3n},$$

$$T_1 = \frac{n(\bar{x}-\theta_1)}{\theta_2},$$

$$T_2 = -\frac{n}{2\theta_2} + \frac{(n-1)s^2 + n(\bar{x}-\theta_1)^2}{2\theta_2^2} - \frac{1}{2n}\left[\frac{n^2(\bar{x}-\theta_1)^2}{\theta_2^2} - \frac{n}{\theta_2}\right] - \frac{2\theta_2}{3n}\left[\left(\frac{\partial L}{\partial\theta_2}\right)^2 - \frac{n}{2\theta_2^2}\right].$$

Now as the crude approximate region is given by

$$\frac{n(\bar{x}-\theta_1)^2}{\theta_2} + \frac{2\theta_2^2}{n}\left(\frac{\partial L}{\partial\theta_2}\right)^2 = \chi_0^2,$$

we may write in the next approximation

$$\begin{aligned} T_2 &= -\frac{n}{2\theta_2} + \frac{(n-1)s^2}{2\theta_2^2} + \frac{1}{2\theta_2} + \frac{1}{3\theta_2} - \frac{1}{3\theta_2}\left[\chi_0^2 - \frac{n(\bar{x}-\theta_1)^2}{\theta_2}\right] \\ &= \frac{n-1}{2\theta_2^2}[s^2-\theta_2] + \frac{1}{3\theta_2}\left[1 + \frac{n(\bar{x}-\theta_1)^2}{\theta_2} - \chi_0^2\right], \end{aligned} \tag{8}$$

obtaining as the next-order approximation for the confidence region

$$\frac{n(\bar{x}-\theta_1)^2}{\theta_2} + \frac{2\theta_2^2}{n}T_2^2 = \chi_0^2, \tag{9}$$

where T_2 is given by (8). As an obvious check on this result, we may note that if we had initially made use of the facts that $\bar{x}$ and s^2 are sufficient independent statistics for θ_1 and θ_2, we should have arrived at practically the same equation (9) by making use of the normal approximation for s^2 developed in I (as noted in I, this is not the *best* normal approximation for s^2, but the one arrived at by the present systematic procedure). The only difference would be that the coefficient of T_2^2 in (9) would be $2\theta_2^2/(n-1)$, this difference arising from the neglect, justifiable to the order of approximation considered, of the alteration in the variance of T_2 due to the third-order moment corrections.

4. The case of 'nuisance parameters'

We come now to the more formidable problem of 'nuisance parameters'. To fix our problem, consider first the case of two parameters θ_1 and θ_2, with θ_2 the nuisance parameter. The

quantity involving $\partial L/\partial\theta_1, \partial L/\partial\theta_2$, which provides an alternative to the maximum-likelihood estimate $\hat{\theta}_1$, is

$$\frac{\partial L}{\partial\theta_1} - \frac{I_{12}}{I_{22}}\frac{\partial L}{\partial\theta_2}, \tag{10}$$

with variance $I_{11} - I_{12}^2/I_{22} \equiv I_{11\cdot 2}$, say. But only in large samples, when the expression in (10) and $\partial L/\partial\theta_2$ become uncorrelated normal variables, with $\partial L/\partial\theta_2$ an effectively sufficient statistic for θ_2, can we expect the standardized variable

$$T \equiv \left(\frac{\partial L}{\partial\theta_1} - \frac{I_{12}}{I_{22}}\frac{\partial L}{\partial\theta_2}\right)\Big/\sqrt{I_{11\cdot 2}} \tag{11}$$

to be completely insensitive to the estimate we substitute for θ_2. In smaller samples this substitution will affect the distribution of T. In principle this effect can be successively allowed for, this being the proposed procedure; the effect of the correction at one stage may be reconsidered, a further correction of a higher order introduced, and so on. As a general method, this is, however, evidently going to become rather intractable after the first stage, for the previous advantages of working directly with the derivatives $\partial L/\partial\theta_1$, $\partial L/\partial\theta_2$ are in general lost.

It is not possible to ignore the sufficiency properties, if any, that exist for θ_2. For if $\partial L/\partial\theta_2$ is equivalent, even in small samples, to a single sufficient statistic for θ_2 (when θ_1 is assumed known), we *can** proceed from T by building up a polynomial series solution in $\partial L/\partial\theta_1$ and $\partial L/\partial\theta_2$ which is *statistically* independent of $\partial L/\partial\theta_2$ and hence *mathematically* independent of θ_2. But if $\partial L/\partial\theta_2$ has not got such sufficiency properties, it can only be considered 'sufficient' for θ_2 if regarded as a random *function* in the parameter θ_2, and statistical independence merely of the random *variable* $\partial L/\partial\theta_2$ would appear to be no longer adequate. I thought at one stage that this difficulty might be overcome by eliminating also the dependence on higher derivatives such as $\partial^2 L/\partial\theta_2^2$, but this may be shown (by a counter-example, see § 6) to be impossible in general. In the general case it seems therefore necessary to fall back on the very cumbersome substitution procedure referred to in the first paragraph above.

Fortunately this substitution, if the maximum-likelihood estimate of θ_2 for given θ_1 is used, only affects the distribution of the statistic T to the order of approximation $O(1/n)$, where n indicates the sample size, a stage higher than the skewness correction, which is in general of $O(1/\sqrt{n})$. For we may write the maximum-likelihood estimate of θ_2 (for given θ_1) approximately as

$$\theta_2 + \frac{\partial L}{\partial\theta_2}\Big/I_{22},$$

so that

$$T_e \sim T + \frac{\partial L}{\partial\theta_2}\frac{\partial T}{\partial\theta_2}\Big/I_{22}. \tag{12}$$

The *estimated* value of the second term in (12) is of course zero. Its sampling effect on T_e may, however, be investigated. The first correcting term in $E\{(T_e)^r\}$ is

$$\begin{aligned} rE\left\{T^{r-1}\frac{\partial L}{\partial\theta_2}\frac{\partial T}{\partial\theta_2}\Big/I_{22}\right\} &= E\left\{\frac{\partial L}{\partial\theta_2}\frac{\partial T^r}{\partial\theta_2}\Big/I_{22}\right\} \\ &= \frac{\partial}{\partial\theta_2}E\left\{T^r\frac{\partial L}{\partial\theta_2}\Big/I_{22}\right\} - E\left\{T^r\left(\frac{\partial L}{\partial\theta_2}\right)^2\Big/I_{22}\right\} - E\left\{T^r\frac{\partial^2 L}{\partial\theta_2^2}\Big/I_{22}\right\} + E\left\{T^r\frac{\partial L}{\partial\theta_2}\frac{\partial I_{22}}{\partial\theta_2}\Big/I_{22}^2\right\}. \end{aligned}$$

* (Added in proof.) I do not mean to imply that we *should*, for in this case the probability of the sample, given $\partial L/\partial\theta_2$, provides a distribution independent of θ_2 (cf. Bartlett, 1936), and the series expansion, illustrated by § 5 on the t-test, is unnecessary except perhaps for finding approximations.

From the asymptotic independence of T and $\partial L/\partial\theta_2$, it is easy to see that all these terms vanish to the first order (for the second and third, we replace $-\partial^2 L/\partial\theta_2^2$ in the third term by I_{22} to the first order, and couple with the second term). Other asymptotically efficient estimates will be equivalent to the same order and may be used if more convenient; their effect at higher stages would have to be investigated separately.

For reference if required, the next term in the expansion of $\delta\theta_2$ gives

$$\delta\theta_2 \sim \frac{\partial L}{\partial\theta_2}\Big/ I_{22} + \left\{ \frac{\frac{\partial L}{\partial\theta_2}\left[\frac{\partial^2 L}{\partial\theta_2^2} + I_{22}\right]}{I_{22}^2} + \frac{\left(\frac{\partial L}{\partial\theta_2}\right)^2 \frac{\partial^3 L}{\partial\theta_2^3}}{2I_{22}^3} \right\}, \tag{13}$$

whence

$$T_e \sim T + \frac{\partial L}{\partial\theta_2}\frac{\partial T}{\partial\theta_2}\Big/ I_{22} + f\frac{\partial T}{\partial\theta_2} + \frac{1}{2}\left(\frac{\partial L}{\partial\theta_2}\right)^2 \frac{\partial^2 T}{\partial\theta_2^2}\Big/ I_{22}^2, \tag{14}$$

where f is the expression in curly brackets in (13).

A result of some interest which may be derived from an expansion like (13) is

$$E\{\hat{\theta}\} = \theta - \left\{\tfrac{1}{2}E\left(\frac{\partial^3 L}{\partial\theta^3}\right) + \frac{\partial I}{\partial\theta}\right\}\Big/ I^2 + \ldots, \tag{15}$$

where the suffix 2 has been dropped for convenience, as (13) and (15) refer effectively to a single unknown parameter.

While it would be too laborious to carry through the complete details of the general correction to the $O(1/n)$ stage, it may be advisable to indicate how this would proceed, so that the iterative nature of the corrections is made clear. To $O(1/\sqrt{n})$ we have

$$S_e = [T - \tfrac{1}{6}\gamma_1(T^2 - 1)]_e,$$

where γ_1 is the skewness of T. Owing *both* to the properties of S and to the substitution of the estimate S_e for S, so that S_e is related to S by an expansion similar to that for T_e in (14), we have $O(1/n)$ additions to the mean, variance, third and fourth cumulants of, say, c_1, c_2, c_3 and c_4 respectively. Then a new quantity R correct to $O(1/n)$ is (cf. Cornish & Fisher, 1937)

$$R = S_e - c_1 - \tfrac{1}{2}c_2 S_e - \tfrac{1}{6}c_3(S_e^2 - 1) - \tfrac{1}{24}c_4(S_e^3 - 3S_e); \tag{16}$$

the effect of further substitution for θ_2 in c_1, c_2, c_3 and c_4 will not affect the distribution of R_e before at least $O(1/n^{\frac{1}{2}})$.

As noted in I, a statement of general conditions could hardly be expected under which the required monotonic relation existed between our final statistical function and θ_1 if a valid confidence interval is to be possible, but except for very small samples we should not expect these higher-order corrections to disturb any such relation holding in the relevant range; this expectation has been supported by numerical examples.

5. The t-test and the Behrens-Fisher problem

It seems of some interest to demonstrate the procedure first in the case of the standard t-test, even although not only do we not require it here but from the above remarks it does not illustrate the completely general case.

From the expression (6) we obtain for the quantity T defined in (11)

$$T = \frac{\sqrt{n}(\bar{x} - \theta_1)}{\sqrt{\theta_2}}.$$

From the sufficiency property of $\partial L/\partial\theta_2$ in this example, a successive elimination of the higher cumulants between $\partial L/\partial\theta_2$ and the modified series

$$\sqrt{\frac{n}{\theta_2}}T_\mu \equiv \sqrt{\frac{n}{\theta_2}}T + \mu_{20}\left[\left(\frac{\partial L}{\partial\theta_1}\right)^2 - I_{11}\right] + \mu_{11}\left[\frac{\partial L}{\partial\theta_1}\frac{\partial L}{\partial\theta_2} - I_{12}\right] + \mu_{02}\left[\left(\frac{\partial L}{\partial\theta_2}\right)^2 - I_{22}\right] + \dots$$

will eliminate the dependence of T_μ on θ_2. We choose the μ's also so that T_μ becomes normal. The resulting equations are similar to (5), being

$$\kappa_{30} + 6\mu_{20}I_{11}^2 = 0, \quad \kappa_{21} + 2\mu_{11}I_{11}I_{22} = 0, \quad \kappa_{12} + 2\mu_{02}I_{22}^2 = 0,$$

or $\mu_{20} = 0$, $\mu_{11} = -\theta_2/n$, $\mu_{02} = 0$, so to the next approximation stage

$$T_\mu \equiv \frac{\sqrt{n}(\bar{x}-\theta_1)}{\sqrt{\theta_2}}\left(1 - \frac{1}{2}\left\{\frac{(n-1)s^2 + n(\bar{x}-\theta_1)^2}{n\theta_2} - 1\right\}\right)$$

$$\sim \frac{\sqrt{n}(\bar{x}-\theta_1)}{\left\{\dfrac{(n-1)s^2 + n(\bar{x}-\theta_1)^2}{n}\right\}^{\frac{1}{2}}}.$$

This result is not of course yet correct to $O(1/n)$, but there is little point in continuing further with the corrections. If we make use of (12), we reach the same result to this order more simply. Higher approximations would merely develop the Fisher (1941) asymptotic expansion for t.

In the case of the Behrens-Fisher problem of the difference of two independent normal variables with separately unknown variances, we have

$$L = -\tfrac{1}{2}\log(\theta_1+\theta_2) - \frac{1}{2}\frac{(x-\theta_3)^2}{\theta_2+\theta_3} - \frac{n_1}{2}\log\theta_1 - \frac{1}{2}\frac{n_1 v_1}{\theta_1} - \frac{n_2}{2}\log\theta_2 - \frac{1}{2}\frac{n_2 v_2}{\theta_2}, \tag{17}$$

where to correspond with the degrees of freedom n_1, n_2 for the estimated variances v_1, v_2, the unknown variances of x_1, x_2 are taken as θ_1 and θ_2, and θ_3 for the unknown true difference corresponding to $x \equiv x_1 - x_2$.

We easily deduce that

$$T = \frac{x-\theta_3}{\sqrt{(\theta_1+\theta_2)}},$$

and if we substitute the most convenient estimates v_1, v_2 for θ_1, θ_2 (these are not the strict maximum-likelihood estimates for given θ_3, but are asymptotically equivalent as n_1, n_2 increase), we arrive at the usual approximate t-quantity (see, for example, Welch (1947)). To reach a result correct to $O(1/n)$ we should require (i) to correct the kurtosis of T (its skewness is obviously zero), (ii) to make the $O(1/n)$ correction corresponding to the effect of the substitution of v_1, v_2 for θ_1, θ_2. There is little point in carrying this through, for it is clear that it would lead to the next order correction in the asymptotic series solution to this problem proposed by Welch (1947). My impression (with which Dr Welch may not necessarily agree) is that his remarkably ingenious solution to this problem may be logically classified with the further asymptotic expansions based on the normal approximation which are the subject-matter of I and the present paper.

If we alternatively tried to extend the first procedure illustrated for the t-test, to cover two nuisance parameters θ_1 and θ_2, we should eliminate the dependence on both $\partial L/\partial\theta_1$ and $\partial L/\partial\theta_2$. It has been verified to the first stage of the approximation procedure that this leads similarly to a consideration of the approximate t-quantity $(x-\theta_3)/\sqrt{(v_1+v_2)}$, but we would not expect it to extend to higher stages for the reasons given in §4, as $\partial L/\partial\theta_1$ and $\partial L/\partial\theta_2$ do not correspond to a pair of sufficient statistics for θ_1 and θ_2 (for given θ_3).

6. Fisher's analysis of variance problem

A simpler problem for which such difficulties may be illustrated is the second problem considered by Fisher (1935) in his first derivation of solutions in terms of simultaneous fiducial probability. This problem may be stated for our present purpose as follows. We have two χ^2-quantities

$$\frac{nv}{\theta_1+\lambda\theta_2}, \quad \frac{n_2 v_2}{\theta_2},$$

with degrees of freedom n and n_2, and require a confidence interval for θ_1. Here

$$\begin{aligned} L &= -\frac{n}{2}\log(\theta_1+\lambda\theta_2) - \frac{n}{2}\left(\frac{v}{\theta_1+\lambda\theta_2}\right) - \frac{n_2}{2}\log\theta_2 - \frac{n_2}{2}\frac{v_2}{\theta} \\ &= L(n) + L(n_2), \text{ say}, \\ \frac{\partial L}{\partial\theta_1} &= \frac{n}{2(\theta_1+\lambda\theta_2)^2}[v-(\theta_1+\lambda\theta_2)], \\ \frac{\partial L}{\partial\theta_2} &= \lambda\frac{\partial L}{\partial\theta_1} + \frac{\partial L(n_2)}{\partial\theta_2}, \\ I_{11} &= \frac{n}{2(\theta_1+\lambda\theta_2)^2}, \quad I_{12} = \lambda I_{11}, \quad I_{22} = \lambda^2 I_{11} + \frac{n_2}{2\theta_2^2}, \end{aligned}$$

whence after reduction we obtain

$$T = (v-\lambda v_2-\theta_1)\sqrt{\frac{nn_2}{2[n_2(\theta_1+\lambda\theta_2)^2+\lambda^2 n\theta_2^2]}}. \tag{18}$$

While this problem appears at first sight rather simple, depending on two joint sufficient statistics v and v_2 for θ_1 and θ_2, the difficulty arises from the absence of a sufficient statistic for θ_2 (for given θ_1). Hence we cannot expect it to suffice to modify T to be independent of $\partial L/\partial\theta_2$. In this example we may notice further that it is impossible to make T independent also of, say, $\partial^2 L/\partial\theta_2^2$, for as T, $\partial L/\partial\theta_2$ and $\partial^2 L/\partial\theta_2^2$ are all linear in v and v_2, there is a linear relation between them. Thus we return to the substitution procedure recommended in §4.

It will also be seen in this example that (as in the Behrens-Fisher problem) the use of the normal approximation depends on *two* sample numbers not being too small, in this case n and n_2. The best estimate of θ_2 for given θ_1 is not v_2, but can be obtained from the equation $\partial L/\partial\theta_2 = 0$ using v_2 as a first approximation. This leads to the estimate

$$\left\{\frac{n_2}{v_2}+\frac{\lambda n(v-\theta_1)}{(\theta_1+\lambda v_2)^2}\right\}\Big/\left\{\frac{n_2}{v_2^2}+\frac{\lambda^2 n}{(\theta_1+\lambda v_2)^2}\right\}. \tag{19}$$

We may not substitute the simpler estimate v_2 (unless $n_2 \gg n$), for though its accuracy is of the same order as the estimate in (19), it is not asymptotically independent of T and hence the effect of the substitution will be of $O(1/\sqrt{n}, 1/\sqrt{n_2})$ and must also be corrected even at the first stage of the correction.

The skewness of T, unaffected to the first order by the substitution, is

$$\gamma_1 = 8\left\{\frac{(\theta_1+\lambda\theta_2)^3}{n^2}-\frac{\lambda^3\theta_2^3}{n_2^2}\right\}I_{11\cdot 2}^{\frac{1}{2}}, \tag{20}$$

where $I_{11\cdot 2}$ is the expression under the square root in (18).

Thus corrected to $O(1/\sqrt{n}, 1/\sqrt{n_2})$ we may consider our confidence interval for θ_1 given by either of the two equations (cf. I)

$$T - \tfrac{1}{6}\gamma_1(T^2-1) = \pm\mu, \tag{21a}$$

$$T - \tfrac{1}{6}\gamma_1(\mu^2-1) = \pm\mu, \tag{21b}$$

where in (21*a*) and (21*b*) it is understood that the estimate in (19) is written for θ_2. An equation correct to $O(1/n, 1/n_2)$ would be rather complicated, for in addition to a correction for the substitution for θ_2, a correction for kurtosis would be required; this would make the total correction exceedingly cumbersome, as a bias, variance correction and the asymmetry must all be allowed for at this stage (see §4).

However, from the check made in I on the accuracy of the confidence interval for a single variance, the use of equation (21*a*) or (21*b*) would be not unreasonable for moderate-sized n and n_2.

Table 1

n	χ^2 quantities	θ_2	v	v_2	Lower and upper $P = 0{\cdot}05$ limits for θ_1
12	11·76, 13·43	0	0·980	0	0·55 (0·56), 2·32 (2·25)
		2	1·960	2·240	0, 3·53
		4	2·940	4·476	0, 4·74
	15·24, 5·46	0	1·270	0	0·72 (0·72), 3·00 (2·92)
		2	2·540	0·910	0·91, 5·59
		4	3·810	1·820	0·86, 8·06
6	1·85, 7·80	0	0·308	0	0·14 (0·15), 1·28 (1·13)
		2	0·617	2·600	0, 1·16
		4	0·926	5·196	0, 1·15
	5·64, 4·75	0	0·940	0	0·44 (0·45), 3·92 (3·46)
		2	1·880	1·583	0, 7·09
		4	2·820	3·167	0, 10·32

To illustrate and examine the numerical use of equation (21*b*), it was applied to a few artificial sampling estimates in the case when there are only two observations per group, so that in the analysis of variance table the between-and-within-groups have an equal number of degrees of freedom, n. The true mean square between groups is $2\theta_1 + \theta_2$, so to correspond to the notation above (with $\lambda = \frac{1}{2}$) we take as v one-half of the between-groups mean square and v_2 as the within-groups mean square. Two cases were taken for $n = 12$, and two for $n = 6$.* In addition, the same effective χ^2 quantities were used when comparing the results for two different values of θ_2 (2 and 4), so that the extent to which the inference depended on θ_2 could be more easily seen. On this last point, however, it should be recalled that the statistic used is only independent of our error in estimating θ_2 up to and including $O(1/\sqrt{n})$ in *distribution*, and in any particular case (for which the *error* in θ_2 will alter with θ_2 even for the same χ^2) will still change to an extent of $O(1/\sqrt{n})$.

Moreover, the information on θ_1 is of course less as θ_2 increases, so the width of the interval should in any case tend to increase with θ_2. For comparison the case $\theta_2 = 0$ has been

* It had originally been intended to examine ten cases each, but as this would still be far from sufficient to test the approximation adequately, it was later thought hardly worth while. The sample quantities v and v_2 made use of were partly selected from the larger set as interesting combinations to try out, and thus are not entirely random.

included, as this reduces to the single variance problem discussed in I, and the exact confidence interval in this case can also be given in Table 1 (in brackets). The approximate intervals for $\theta_2 \neq 0$ were obtained from equation (21*b*) graphically, plotting against θ_1 for the lower limit and $1/\theta_1$ for the upper limit (these scales are indicated by the form of the equation at the permissible extreme points $\theta_1 \to 0$ and $1/\theta_1 \to 0$). The true value of θ_1 throughout was unity.

The lower limit for θ_1 will often be the minimum permissible value zero, rather than the value corresponding to $\mu = -1{\cdot}645$, which may well not be reached, especially for small n and large θ_2. This is not unexpected, as it is well known that even the strict maximum-likelihood estimate $v - \lambda v_2$ in this problem may often turn out to be negative; it implies, however, that we are not making full use of the 0·05 risk of error at the lower value, so that it seems useful to think of the error risks for the lower and upper limits separately.*

7. Application to time-series analysis

It has previously been noted (Bartlett, 1952, at end of paper) that the technique may be applied to unknown parameters in time-series. We are now in a position to consider further the solution proposed there for the auto-correlation of a normal Markoff process, in discrete time, and to give a modified solution corrected to $O(1/\sqrt{n})$. The quantity L is, apart from the end-correction (which will not affect our solution until $O(1/n)$)

$$L = -\frac{n}{2}\log\sigma_Y^2 - \frac{1}{2}\sum_{r=1}^{n} Y_r^2/\sigma_Y^2, \tag{22}$$

where $$X_r - \beta X_{r-1} = Y_r.$$

(For simplicity we assume, as before, that $E\{Y\} = 0$. In (22) the observation X_0 has been discarded.) We have†

$$\frac{\partial L}{\partial \beta} = \sum_{r=1}^{n} (X_r - \beta X_{r-1})\, X_{r-1}/\sigma_Y^2,$$

$$I_{11} = \frac{n}{1-\beta^2}, \quad I_{12} = 0, \quad I_{22} = \frac{n}{2\sigma_Y^4},$$

where $\theta_1 \equiv \beta$, $\theta_2 \equiv \sigma_Y^2$. We have further

$$\frac{\partial^3 L}{\partial \beta^3} = 0, \quad \frac{\partial I_{11}}{\partial \beta} = \frac{2n\beta}{(1-\beta^2)^2},$$

whence $$E\left\{\left(\frac{\partial L}{\partial \beta}\right)^3\right\} = \frac{6n\beta}{(1-\beta^2)^2}.$$

We note finally that $$\hat{\sigma}_Y^2 = \sum_{r=1}^{n} X_r^2(1-\beta^2)/n. \tag{23}$$

Hence our approximate confidence interval for β is determined from the equation

$$\frac{\sum_{r=1}^{n} (X_r - \beta X_{r-1})\, X_{r-1}/\hat{\sigma}_Y^2}{\sqrt{\{n/(1-\beta^2)\}}} - \frac{\beta}{\sqrt{\{(1-\beta^2)\,n\}}}(\mu^2 - 1) = \pm\mu. \tag{24}$$

* In the case $n = n_2$ the expression in (21) at $\theta_1 = 0$ is $\sqrt{n}(v - \lambda v_2)/(v + \lambda v_2)$, which must be greater than μ for a non-zero lower limit. When $\lambda v_2/v < \frac{1}{2}$, it may be shown that the monotonic relation with θ_1 breaks down near $\theta_1 = 0$, but of course *if* θ_1 is near zero $\lambda v_2/v \sim 1 + O(1/\sqrt{n})$, so that no difficulty should arise within the 'admissible' range of θ_1 except perhaps for very small n.

† In equation (25) of the paper cited, σ_Y^2 was inadvertently omitted, but this did not affect the deduced confidence interval equation (26).

It is interesting to notice the appearance of a non-zero skewness correction, in contrast with the classical regression problem.

When we are able to assume normal residuals (or some other complete specification of the process) similar methods can clearly be applied in other time-series problems.

8. Relation with goodness of fit tests

It was pointed out in §3 that an alternative to the approximate χ^2 expression in $\partial L/\partial\theta_i$ would be to use $-2(L-L_{\max.})$, where $L_{\max.}$ denotes the maximum of L for choice of the unknown θ_i. This last expression is more familiar in the use of the χ^2 test as a goodness of fit of hypothetical values of the θ_i, but the two uses are of course closely connected. In this final section we shall demonstrate *to the next order of approximation* ($O(1/\sqrt{n})$) the asymptotic equivalence (in distribution) of this alternative χ^2 approximation,* equivalent to the likelihood ratio λ and given by

$$-2\log\lambda = -2(L-L_{\max.}). \tag{25}$$

Whether we make use of λ or the general quadratic formula

$$\chi^2 \sim I^{ij}\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}, \tag{26}$$

where in (26) I^{ij} denotes the inverse of I_{ij} (the summation convention is used throughout this section), will thus to this order be a matter of convenience. Formula (25) is formally simpler than (26), but involves finding the simultaneous maximum-likelihood estimates $\hat\theta_i$.

We have (expanding $L(\hat\theta_i)$ about $L(\theta_i)$, rather than, as is perhaps more usual, $L(\theta_i)$ about $L(\hat\theta_i)$)

$$L(\hat\theta_i) = L(\theta_i) + \left\{(\hat\theta_i-\theta_i)\frac{\partial L}{\partial\theta_i} + \tfrac{1}{2}(\hat\theta_i-\theta_i)(\hat\theta_j-\theta_j)\frac{\partial^2 L}{\partial\theta_i\partial\theta_j}\right\}$$
$$+\tfrac{1}{6}(\hat\theta_i-\theta_i)(\hat\theta_j-\theta_j)(\hat\theta_k-\theta_k)\frac{\partial^3 L}{\partial\theta_i\partial\theta_j\partial\theta_k}+\ldots. \tag{27}$$

The relation between $\hat\theta_i$ and L may be developed analogously to the case of one unknown quoted in §4 (equation (13)). We have

$$0 = \frac{\partial L}{\partial\theta_i} + (\hat\theta_j-\theta_j)\frac{\partial^2 L}{\partial\theta_i\partial\theta_j} + \tfrac{1}{2}(\hat\theta_j-\theta_j)(\hat\theta_\kappa-\theta_\kappa)\frac{\partial^3 L}{\partial\theta_i\partial\theta_j\partial\theta_\kappa}+\ldots$$
$$= \frac{\partial L}{\partial\theta_i} - (\hat\theta_j-\theta_j)I_{ij} + \left\{(\hat\theta_j-\theta_j)\left(\frac{\partial^2 L}{\partial\theta_i\partial\theta_j}+I_{ij}\right) + \tfrac{1}{2}(\hat\theta_j-\theta_j)(\hat\theta_k-\theta_k)\frac{\partial^3 L}{\partial\theta_i\partial\theta_j\partial\theta_k}\right\}+\ldots.$$

Denote the inverse of I_{ij} by I^{ij}. Then

$$(\hat\theta_k-\theta_k) = I^{ik}\frac{\partial L}{\partial\theta_i} + \left\{\frac{\partial L}{\partial\theta_h}\left[\frac{\partial^2 L}{\partial\theta_i\partial\theta_j}+I_{ij}\right]I^{hj}I^{ik} + \tfrac{1}{2}I^{ik}I^{hj}I^{gm}\frac{\partial L}{\partial\theta_h}\frac{\partial L}{\partial\theta_g}\frac{\partial^3 L}{\partial\theta_i\partial\theta_j\partial\theta_m}\right\}+\ldots$$
$$= I^{ik}\frac{\partial L}{\partial\theta_i} + f_k, \tag{28}$$

say, where

$$f_k = \left\{\frac{\partial L}{\partial\theta_h}\left[\frac{\partial^2 L}{\partial\theta_i\partial\theta_j}I^{hj}I^{ik}+I^{hk}\right] + \tfrac{1}{2}I^{ik}I^{hj}I^{gm}\frac{\partial L}{\partial\theta_h}\frac{\partial L}{\partial\theta_g}\frac{\partial^3 L}{\partial\theta_i\partial\theta_j\partial\theta_m}\right\}+\ldots.$$

* For a discussion of the large-sample χ^2 approximation for the likelihood ratio λ (including the more general case where L is replaced by its maximum $L'_{\max.}$ with respect to a smaller subclass of the unknown parameters), see Wilks (1938), where it is shown that $-2\log\lambda$ is equivalent to a χ^2 except for terms of $O(1/\sqrt{n})$.

Hence

$$-2(L-L_{\max.}) = I^{ik}\frac{\partial L}{\partial\theta_k}\frac{\partial L}{\partial\theta_i}+\left\{2f_i\frac{\partial L}{\partial\theta_i}+I^{ki}\frac{\partial L}{\partial\theta_k}I^{hj}\frac{\partial L}{\partial\theta_h}\left(\frac{\partial^2 L}{\partial\theta_i\partial\theta_j}+I_{ij}\right)\right.$$
$$\left.+2I^{ki}\frac{\partial L}{\partial\theta_k}f_i\frac{\partial^2 L}{\partial\theta_i\partial\theta_j}+\tfrac{1}{3}I^{ki}\frac{\partial L}{\partial\theta_k}I^{hj}\frac{\partial L}{\partial\theta_h}I^{gm}\frac{\partial L}{\partial\theta_g}\frac{\partial^3 L}{\partial\theta_i\partial\theta_j\partial\theta_m}\right\}+\ldots,$$

or if we put $\partial^2 L/\partial\theta_i\partial\theta_j \sim -I_{ij}$ in the fourth term on the right, this may be written

$$I^{ik}\frac{\partial L}{\partial\theta_k}\frac{\partial L}{\partial\theta_i}+\left\{\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}\left(I^{ik}I^{jh}\frac{\partial^2 L}{\partial\theta_k\partial\theta_h}+I^{ij}\right)+\frac{1}{3}\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}\frac{\partial L}{\partial\theta_k}\frac{\partial^3 L}{\partial\theta_g\partial\theta_h\partial\theta_m}I^{ig}I^{jh}I^{km}\right\}+\ldots$$
$$= I^{ik}\frac{\partial L}{\partial\theta_k}\frac{\partial L}{\partial\theta_i}+R, \tag{29}$$

say. Now as the skewness of $\partial L/\partial\theta_k$ does not affect the distribution of the quadratic first term, it is, even to $O(1/\sqrt{n})$, a valid χ^2 quantity, as we saw in §3. It remains to consider the effect of the remainder term R. Now although R is of $O(1/\sqrt{n})$, we can readily show that $E\{R\}$, and $E\{I^{ik}\partial L/\partial\theta_k\partial L/\partial\theta_i R\}$ are of $O(1/n)$, so that $-2(L-L_{\max.})$ has the same mean and variance as $I^{ik}\partial L/\partial\theta_k\partial L/\partial\theta_i$ to $O(1/\sqrt{n})$. The same method can be applied also for the higher moments. To this order the same χ^2 approximation may thus be used. Although these average values involve higher-order moments of L derivatives than we have listed, for our immediate purpose we do not require their explicit values. We merely note that

$$E\left\{\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}\left(\frac{\partial^2 L}{\partial\theta_k\partial\theta_h}+I_{kh}\right)\right\}\leqslant O(n),$$

$$E\left\{\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}\frac{\partial L}{\partial\theta_k}\frac{\partial^3 L}{\partial\theta_g\partial\theta_h\partial\theta_m}\right\}\leqslant O(n^2),$$

$$E\left\{\left(\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}\right)\frac{\partial L}{\partial\theta_g}\frac{\partial L}{\partial\theta_m}\left(\frac{\partial^2 L}{\partial\theta_k\partial\theta_h}+I_{kh}\right)\right\}\leqslant O(n^2),$$

$$E\left\{\left(\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}\right)\frac{\partial L}{\partial\theta_g}\frac{\partial L}{\partial\theta_m}\frac{\partial L}{\partial\theta_p}\frac{\partial^3 L}{\partial\theta_k\partial\theta_h\partial\theta_q}\right\}\leqslant O(n^3),$$

and as I^{ij} is $O(n^{-1})$, the result follows. It is obvious from particular examples that $-2(L-L_{\max.})$ *does* require modification to $O(1/n)$ (so also does the alternative quadratic expression). In principle a general correction to $O(1/n)$ could be worked out by the above methods, though we have seen that corrections at this order are rather formidable to evaluate by the present general approach. Particular problems, e.g. in time-series analysis, might be somewhat more tractable if handled individually.

By an obvious extension of the argument, the criterion $-2(L'_{\max.}-L_{\max.})$, where $L'_{\max.}$ refers to a smaller subclass of the unknown parameters than $L_{\max.}$, is equivalent in distribution to a χ^2, also up to and including $O(1/\sqrt{n})$.

I am indebted to Mrs A. Linnert for assistance with the computations for Table 1.

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* (*Added in proof.*) Professor J. W. Tukey has just sent me a copy of his unpublished (1949) *Memorandum Report* No. 26, *Statistical Research Group, Princeton University*, on 'Standard confidence points', in which similar proposals for a single unknown parameter were put forward.

NON-NORMALITY AND TESTS ON VARIANCES

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1. INTRODUCTION

For many experimenters the most commonly used statistical tests are those for *comparing* sample means and sample variances. The test used for the equality of the means of k groups of observations is usually the analysis of variance test (or equivalently the t test when $k = 2$), whilst the test used for the equality of variances of k groups of observations is Bartlett's (1937) modification of the Neyman-Pearson (1931) L_1 test (or the equivalent F test when $k = 2$).

The tests mentioned are derived on a number of assumptions, in particular, that the observations are normally distributed. Usually, however, since little is known of the populations from which the samples are drawn, these tests are used, of necessity, as if the assumption of normality could be ignored. So far as comparative tests on means are concerned it appears (perhaps rather surprisingly) that this practice is largely justifiable, for thanks to the work of Pearson (1931), Bartlett (1935), Geary (1947), Gayen (1950 *a*, *b*), David & Johnson (1951 *a*, *b*) there is abundant evidence that these *comparative* tests on means are remarkably insensitive to general* non-normality of the parent population.

It would appear, however, that this remarkable property of 'robustness' to non-normality which these tests for comparing means possess, and without which they would be much less appropriate to the needs of the experimenter, is not necessarily shared by other statistical tests, and in particular is not shared by the tests for equality of variances, mentioned above.

The sensitivity to non-normality of the tests for comparing *two* variances was first pointed out by E. S. Pearson (1931) whose findings were confirmed by Geary (1947), Finch (1950) and Gayen (1950*a*). These authors showed that this test is particularly sensitive to changes in γ_2 from the normal theory value of zero. (The notation γ_1, γ_2, etc., will be used for the standardized parent cumulants. In Karl Pearson's notation $\gamma_1 = \sqrt{\beta_1}$, $\gamma_2 = \beta_2 - 3$.) In the present paper it is shown that the sensitivity is even greater when the number of variances to be compared exceeds two, and, indeed, that the sensitivity to non-normality of the L_1 criterion or the equivalent Bartlett test can be of the same order of magnitude as the sensitivity of criteria such as b_2 specifically designed to *test* normality.

It is further shown that the difference in sensitivity of the test on means on the one hand and the test on variances on the other arises from an essential difference in the nature of these two types of tests and this suggests how the extreme sensitivity to non-normality of the variance tests can be remedied.

* By 'general' parent non-normality is meant that the departure from normality, in particular skewness, is the same in the different groups, as could usually be assumed when the data were from an experiment in which the groups corresponded with different applied treatments to be compared. In tests in which sample means are *compared*, general skewness tends to be cancelled out; larger effects are found, however, if the skewness is in different directions in the different groups.

2. Distribution of variance test for large samples

Suppose there are k groups of independently distributed observations with n_t in the tth group and $\sum_t n_t = N$. The ith observation of the tth group is denoted by y_{it} and the usual estimate of variance in the tth group, having $\phi_t = n_t - 1$ degrees of freedom, by s_t^2. The average of the estimated variances is denoted by s^2, where $s^2 = \sum_t \phi_t s_t^2/\Phi$ and Φ is the total number of degrees of freedom $\Sigma\phi_t = \Phi$.

The L_1 criterion due to Neyman & Pearson (1931) tests the hypothesis that the *variances* in the groups are all equal, given that the observations are normally distributed. Alternatively, M_1, the logarithmic modification due to Bartlett (1937), may be employed:

$$M_1 = \Phi \ln s^2 - \sum_t \phi_t \ln s_t^2. \tag{1}$$

Following Neyman & Pearson (1931) it may be shown that when the appropriate null hypothesis is true and on the assumption of parent normality, M_1 is distributed in large samples as χ^2_{k-1} (where χ^2_{k-1} means χ^2 with $k-1$ degrees of freedom). This is the basis of the test proposed by Bartlett (1937). He shows that, even for fairly small samples, to a close approximation M_1 is distributed as $(1+A)\chi^2_{k-1}$, where A, an adjustable constant which tends to zero for large values of ϕ_t, is given by

$$A = \frac{1}{3(k-1)}\left\{\sum_t \frac{1}{\phi_t} - \frac{1}{\Phi}\right\}. \tag{2}$$

It is convenient to define the quantity

$$d = \left(1 + \frac{1}{2}\frac{\phi}{\phi+1}\gamma_2\right)^{-1}. \tag{3}$$

When ϕ tends to infinity this becomes

$$\delta = (1 + \tfrac{1}{2}\gamma_2)^{-1}. \tag{4}$$

Thus δ is an alternative measure of kurtosis and the quantities β_2, γ_2 and δ are related as follows:

β_2	1	2	3	6	∞
γ_2	-2	-1	0	3	∞
δ	∞	2	1	0·4	0

Denote by $s^2_\phi(\gamma_2)$ an estimate of variance having ϕ degrees of freedom and based on observations drawn from a population with finite cumulants whose measure of kurtosis is γ_2. Then $s^2_\phi(0)$ is an estimate from a normal population. The estimate $s^2_\phi(\gamma_2)$ has mean σ^2 and variance $2\sigma^4/(d\phi)$. Also from the general expressions for the higher cumulants of such an estimate it is seen that as ϕ tends to infinity the distribution of $s^2_\phi(\gamma_2)$ tends to normality, and for sufficiently large values of ϕ, $s^2_\phi(\gamma_2)$ is distributed like $s^2_{d\phi}(0)$.

Denote by $M_1(0; \phi_1\phi_2, \ldots, \phi_k)$ the criterion calculated from normally distributed observations and by $M_1(\gamma_{21}\gamma_{22}\ldots\gamma_{2k};\ \phi_1\phi_2\ldots\phi_k)$ the criterion calculated from k groups of observations drawn from populations with measures of kurtosis $\gamma_{21}, \gamma_{22}, \ldots, \gamma_{2}\ldots\gamma_{2k}$. Then for large samples

$$M_1(\gamma_{21}\gamma_{22}\ldots\gamma_{2k};\ \phi_1\phi_2\ldots\phi_k)$$

is distributed as

$$\Phi \ln\{\textstyle\sum \phi_t s^2_{d_t\phi_t}(0)/\Phi\} - \sum_t \phi_t \ln s^2_{d_t\phi_t}(0). \tag{5}$$

When the samples are sufficiently large we may write δ_t for d_t, and in the important case where the kurtosis in each of the sampled populations is the same, so that $\delta_t = \delta$ for all t, $M_1(\gamma_2; \phi_1 \phi_2 \dots \phi_k)$ is distributed as

$$\delta^{-1}\{\delta\Phi \ln s^2_{\delta\Phi}(0) - \sum_t \delta\phi_t \ln s^2_{\delta\phi_t}(0)\} = \delta^{-1} M_1(0; \delta\phi_1, \delta\phi_2, \dots, \delta\phi_k). \quad (6)$$

Consequently $M_1(\gamma_2)$ is distributed asymptotically not as χ^2_{k-1} but as $(1 + \frac{1}{2}\gamma_2)\chi^2_{k-1}$ for any parent distribution having finite cumulants. We see therefore that M_1 is asymptotically biased if γ_2 is not zero. In particular, in large samples the mean of the distribution curve of M_1 would be $(1 + \frac{1}{2}\gamma_2)(k-1)$ instead of $k-1$ and the standard deviation $(1 + \frac{1}{2}\gamma_2)\{2(k-1)\}^{\frac{1}{2}}$ instead of $\{2(k-1)\}^{\frac{1}{2}}$. The discrepancy in means relative to the standard deviation would thus become larger as k was increased. Consequently divergences from normal theory probabilities, known to be large when $k = 2$, would become progressively larger for $k > 2$. Table 1 shows the true probability of exceeding the normal-theory 5 % significance levels in large samples for various values of k and γ_2 when the null-hypothesis is true.

Table 1. *True percentage chance of exceeding 5 % normal-theory significance level of M_1 in large samples from non-normal populations for various values of γ_2*

γ_2	No. of groups = k					
	2	3	5	10	20	30
2	16·6	22·4	31·5	48·9	71·8	84·9
1	11·0	13·6	17·6	25·7	38·9	49·8
0	5·0	5·0	5·0	5·0	5·0	5·0
−1	0·56	0·25	0·08	0·010	0·0004	0·00001

In the case when the null-hypothesis is not true it will be seen that changes in the level of the criterion due to non-normality and due to real differences in the group variances will tend to be 'confounded'. With leptokurtic populations the criterion will tend to show differences when none exist, while with platykurtic populations real differences will tend to be masked.

The behaviour of the test for means is in sharp contrast to that for variances discussed above. The appropriate test for the hypothesis—that the *means* in the groups are all equal, given that the variances are all equal, and that the observations are normally distributed—is the 'within and between groups' analysis of variance test. In its commonest form the criterion used is

$$F = \text{(between groups mean square)/(within groups mean square)}$$
$$= \left\{\sum_t \frac{n_t(\bar{y}_t - \bar{y})^2}{k-1}\right\} \Big/ s^2.$$

Alternatively, to correspond with M_1, we could employ a form obtained from the logarithm of the likelihood ratio

$$M_2 = (N-1)\ln\{1 + (k-1)F/(N-k)\}.$$

For large samples Neyman & Pearson (1931) showed that assuming normality a criterion equivalent to M_2 would be distributed as χ^2_{k-1}. With certain not very severe restrictions on the parent population it is possible to show that this result is true *whether the population is*

normal or not. Thus for large samples the criterion M_2 for means tends to follow the normal-theory distribution under conditions of non-normality. This explains to some extent the insensitivity to non-normality found by Pearson and other workers mentioned above. In contrast, the criterion M_1 for variances follows a distribution directly dependent on γ_2 however large are the samples.

It is perhaps unfortunate that the test for homogeneity of means, and the test for homogeneity of variances in the particular case of two groups, can each be brought to the form of a variance ratio test. This has sometimes led to apparently contradictory statements about the sensitivity of the 'variance ratio' to non-normality. As originally pointed out by Pearson in 1931, the two criteria are really essentially different in character but follow the same distribution when the parent is normal. The marked difference in sensitivity of the criteria is well brought out by an example taken from Gayen's work (1950*a*). A test for homogeneity of *means* of five groups of five observations leads to an F test on 4 and 20 degrees of freedom; on the normal assumption this will be distributed in the same form as the F criterion used to compare the *variances* in two independent groups of 5 and 21 observations. Gayen shows, for example, that for a certain type of non-normal distribution in which $\gamma_1 = 0$ and $\gamma_2 = 2$ the true probability of exceeding the 5 % point for the test on means would be 4·5 % (a quite trivial discrepancy), whilst that for the test on variances would be 10·2 %.

3. Distribution of M_1 in small samples for certain non-normal parent distributions

It will now be shown that parent distributions exist for which relations similar to (6) are obtained even for small samples.

(3·1) *Population means known*

For the moment it will be assumed that the means of the k sampled populations are known. Denote by $s^2_{.n}$ an estimate of variance having n degrees of freedom calculated from the n deviations from the known group mean η. The criterion calculated from such estimates is denoted by $M_{.1}$. If the distribution of the observations is such that $s^2_{.n}(\gamma_2)$ is distributed like $s^2_{.\delta n}(0)$ for any sample size then $ns^2_{.n}(\gamma_2)$ is distributed like $\delta^{-1}\chi^2_{\delta n}\sigma^2$, which implies that $(y-\eta)^2$ is distributed like $\delta^{-1}\chi^2_\delta\sigma^2$. This is true if

$$p(y) = (\delta/2\sigma^2)^{\frac{1}{2}\delta}\{\Gamma(\tfrac{1}{2}\delta)\}^{-1}\,|\,y-\eta\,|^{\delta-1}\exp\{-\delta(y-\eta)^2/2\sigma^2\} \quad (-\infty<y<+\infty;\ 0<\delta<\infty). \tag{7}$$

This is a double χ distribution, that is to say, it is a symmetrical distribution of the form of two χ distributions having δ degrees of freedom 'back to back'. It has mean η, variance σ^2, $\gamma_1 = 0$ and $\gamma_2 = 2(1-\delta)/\delta$ (that is, $\delta^{-1} = 1+\frac{1}{2}\gamma_2$ as previously defined). When $\delta = 1$ the term in $|\,y-\eta\,|$ vanishes and the distribution is normal. When δ is less than 1, γ_2 is greater than zero, and a leptokurtic distribution results. When δ is greater than 1, γ_2 is less than zero and a bimodal platykurtic distribution is obtained. For such parent distributions it is easily seen by the previous argument that

$$M_{.1}(\gamma_2;\ n_1, n_2, \ldots, n_k) \text{ is distributed } \textit{exactly} \text{ as } \delta^{-1}M_{.1}(0;\ \delta n_1, \delta n_2, \ldots, \delta n_k), \tag{8}$$

where $n_1, n_2, \ldots, n_k$ refer to the sample sizes.

(3·2) *Population means unknown*

In 1931 Le Roux carried out a very painstaking research into the distribution of the variance calculated from small samples drawn from non-normal distributions of Pearson

type. He studied changes in the distribution of s^2 as the parent distribution changed, by calculating its first four moments from the moments of the parent distribution, and confirmed his findings by extensive sampling experiments. One of his discoveries was that in addition to the normal distribution a series of non-normal distributions existed (which he called $D(s^2)$ III distributions) for which the first four moments of s^2 were almost exactly those of a Pearson type III curve. These $D(s^2)$ III distributions show kurtosis accompanied by marked skewness. A selection of these adapted from Le Roux's table are shown below.

$D(s^2)$ III distributions

γ_1^2	0·00	0·10	0·20	0·40	0·60	0·80	1·00	1·20	1·40
γ_2	0·00	−0·04	0·03	0·24	0·48	0·73	0·99	1·25	1·51

For such curves Le Roux deduced that to a close approximation $s^2_\phi(\gamma_2)$ was distributed as $s^2_{d\phi}(0)$ for any sample size $\left(\text{where as before } d^{-1} = 1+\frac{1}{2}\frac{\phi}{\phi+1}\gamma_2\right)$, a conclusion supported by his sampling experiments. We may conclude therefore that the relation (5) is approximately true for samples of any size chosen from such populations.

In particular, if the degree of kurtosis is the same in each of the populations sampled and each variance is based on the same number ϕ of degrees of freedom, then for $D(s^2)$ III populations

$$M_1(\gamma_2;\phi,\phi,\ldots,\phi) \text{ is distributed approximately as } d^{-1}M_1(0;d\phi,d\phi,\ldots,d\phi).$$

Using Bartlett's method we see that for $D(s^2)$ III populations therefore $M_1(\gamma_2;\phi_1,\phi_2,\ldots,\phi_k)$ is distributed approximately not as $(1+A)\chi^2_{k-1}$ but as

$$d^{-1}(1+Ad^{-1})\chi^2_{k-1}, \tag{9}$$

where A is the correcting factor of order ϕ^{-1} defined in (2). We note in passing that departure from normality of this form would not need to be very great or the sample size very big before the effect of the constant d^{-1} completely swamped the effect of the correction term A.

4. Two groups of observations

When $k = 2$ we may use the variance ratio F as an alternative form of the M_1 criterion. Using the same argument as before and a corresponding notation it is apparent that for samples of any size drawn from double χ distributions with measures of kurtosis γ_{21} and γ_{22},

$$F(\gamma_{21},\gamma_{22};n_1,n_2) \text{ is distributed exactly as } F(0;\delta_1 n_1,\delta_2 n_2), \tag{10}$$

whilst for samples of any size from $D(s^2)$ III populations

$$F(\gamma_{21},\gamma_{22};\phi_1,\phi_2) \text{ is distributed approximately as } F(0;d_1\phi_1,d_2\phi_2). \tag{11}$$

These results, which are, of course, equivalent asymptotically, would be expected to apply for *any population* when the sample sizes were sufficiently large. To provide some indication of how large the samples would have to be, comparison was made with the results given by (11) and the more exact values calculated by other authors. Gayen (1950*b*) and Finch (1950) have considered the distribution of the variance ratio or the equivalent z criterion for populations defined by the first few terms of Edgeworth's series. Both these authors note that for these populations the effect of changes in γ_1 is small but large discrepancies are produced by changes in γ_2. In Fig. 1, γ_1 is assumed zero and the continuous lines represent the probability as determined by these authors, of exceeding the normal theory 5 % point

for various values of γ_2. The values of γ_2 are assumed to be the same in each of the two populations. The first graph illustrates the case $\phi_1 = 24$, $\phi_2 = 60$ considered by Finch, and the second the case $\phi_1 = 4$, $\phi_2 = 20$ considered by Gayen. The dotted lines are the values given by equation (11) above. A further comparison is supplied by Pearson's (1931) sampling experiments. Pearson drew 500 pairs of samples of 5 and 20 from six experimental populations and calculated a criterion equivalent to F; this corresponds very nearly to the case $\phi_1 = 4, \phi_2 = 20$ above (ϕ_2 is actually 19 instead of 20). In his table Pearson shows the number of pairs of samples for which the criterion exceeds the 4·86 % point (which again is very close to the 5 % point considered above). These numbers reduced to percentages are plotted as circles about the second line.

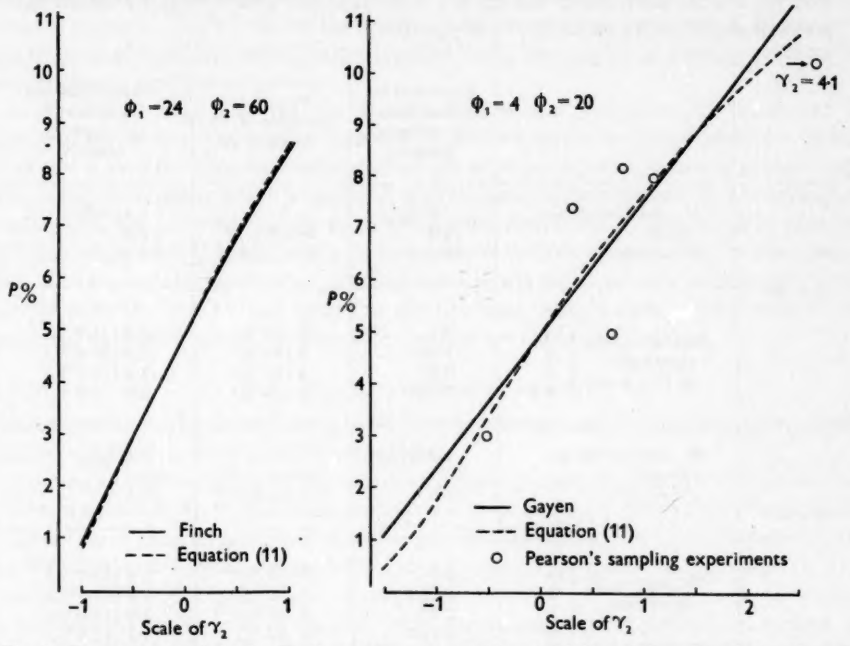

Fig. 1. Percentage probability P of exceeding normal-theory 5 % level plotted against value of γ_2.

The agreement between the simple approximation (11) and the values found by Finch is very close, as would be expected for the comparatively large sample sizes used. The extent of agreement with Gayen's results is rather surprising when the small size of one of the samples is remembered. In both approaches it is assumed, however, that the higher cumulants are finite. The marked departure from both lines which occurs when this is not so is shown by Pearson's sampling experiment from the type VII (i.e. the Student t) distribution for which $\gamma_2 = 4{\cdot}1$ and higher cumulants are infinite.

5. Sampling experiments with a $D(s^2)$ III distribution

Confirmation of (9) in the case $k = 20$ was obtained from an experiment in which samples were drawn from a population nearly of type $D(s^2)$ III form. Le Roux drew 1000 samples of $n = 20$ observations from such a distribution in which $\gamma_1^2 = 1{\cdot}0$ and $\gamma_2 = 0{\cdot}8$. (The dis-

tribution was actually a Pearson type I.) He calculated the quantity $(n-1)s^2/n$ for each of the 1000 samples and published the resulting frequency distribution. The samples were grouped but the grouping interval was fairly fine. In the present work Le Roux's observed distribution of s^2 has been resampled, the sampling being without replacement. In each experiment twenty-five groups of twenty variance estimates were reconstructed and from these, twenty-five values of M_1 were calculated. The experiment was then repeated using Le Roux's observed distribution of variance estimates for samples of $n = 5$ from the same parent distribution.

Table 2. *Characteristics of observed distributions of twenty-five values of M_1 for $k = 20$ groups, with (a) $n = 20$ observations and (b) $n = 5$ observations, drawn from a Pearson type I parent distribution for which $\gamma_1^2 = 1{\cdot}0$, $\gamma_2 = 0{\cdot}8$*

		Expected on normal theory (χ^2 approximation)	Found in samples of 25 values of M_1	Expected from equation (9) (χ^2 approximation)
$n = 20$	M_1 distributed as:	$1{\cdot}018\chi^2_{19}$		$1{\cdot}415\chi^2_{19}$
	Mean	19·4	28·6 ± 1·9*	26·9
	Variance	39·4	93·3 ± 30·3†	76·1
	No. out of 25 significant at P % point: P			
	10	2·5	13 (52 %)	10·3 (41·2 %)
	5	1·25	9 (36 %)	7·5 (30·0 %)
	1	0·25	4 (16 %)	3·2 (12·8 %)
	0·1	0·025	1 (4 %)	0·9 (3·6 %)
$n = 5$	M_1 distributed as:	$1{\cdot}088\chi^2_{19}$		$1{\cdot}473\chi^2_{19}$
	Mean	20·7	28·0 ± 1·8*	28·0
	Variance	44·9	79·1 ± 25·6†	82·4
	No. out of 25 significant at P % point: P			
	10	2·5	13 (52 %)	9·6 (38·2 %)
	5	1·25	9 (36 %)	6·8 (27·1 %)
	1	0·25	2 (8 %)	2·8 (11·0 %)
	0·1	0·025	0 (0 %)	0·7 (2·9 %)

* The quantity following the ± signs is the approximate standard error of the mean obtained from S.E.(Mean M_1) $\simeq \{V(M_1)/25\}^{\frac{1}{2}}$.

† The quantity following the ± signs is a rough estimate of the standard error of the variance of M_1 obtained by assuming that M_1 is approximately distributed in Pearson type III form and substituting $\gamma_2 = 12/(k-1)$ in the approximate formula, S.E.$\{V(M_1)\} \simeq V(M_1)\{(\gamma_2+2)/25\}^{\frac{1}{2}}$.

The two observed distributions of M_1 are shown in the sixth and seventh columns of the table in the Appendix. The main features of the observed distributions are shown in Table 2. It will be seen that the means and variances of the values of M_1 are in close agreement with the values anticipated from equation (9) and differ very markedly from those expected on normal theory. The excess of 'significant' values is also verified. For example, both for samples of $n = 20$ and $n = 5$ observations, nine out of twenty-five (or 36 %) of the values were significant at the 5 % point.

6. Sampling experiments with other distributions

Besides the population of the $D(s^2)$ III type, Le Roux considered a variety of other populations, mostly using sampling results used by E. S. Pearson in earlier investigations, to determine empirically the distribution of the sample variance. In the present investigation Le Roux's distributions derived from the following parent distributions have been resampled in the manner already described:

Pearson type	II	III	I	IV	VII
(γ_1^2, γ_2)*	(0·0, −0·5)	(0·5, 0·75)	(1·0, 0·8)	(1·2, 2·8)	(0·0, 4·1)

From each distribution, twenty-five sets of twenty samples of s^2 have been obtained, and for these twenty-five values of M_1 calculated. These observed values of M_1 are set out in the Appendix. Table 3 shows the values of the means and variances of M_1 observed and those expected on normal theory, together with the number of values exceeding the normal theory significance points.

It is seen that large discrepancies from the normal theory distributions occur, and the discrepancies are in the directions expected. The discrepancies are larger when $n = 20$ than when $n = 5$, and for the populations studied the asymptotic result seems to give an upper limit to the discrepancy which is approached as the sample size is increased. The asymptotic result appears to be more rapidly approached when kurtosis is accompanied by skewness till for the skew $D(s^2)$ III populations, the asymptotic value is attained even for small sample sizes. The asymptotic result is approached more slowly when kurtosis is large and for the very leptokurtic type IV and type VII distributions, sample sizes greater than twenty appear to be needed before the asymptotic values are closely approached.

7. M_1 as a test for normality

The seriousness of the discrepancies to be expected, even in small samples, may be appreciated by considering M_1 as a *test* for normality.

Suppose a sample of N observations is drawn from a population distributed in the double χ form of equation (7). We shall assume that the mean η of the population is known, so that without loss of generality we may take it to be zero. We wish to test the hypothesis H_0 that the distribution is normal, i.e. that $\delta = 1$, $\sigma > 0$, against the class of alternatives H_1 which specify that the distribution is not normal, i.e. that $\delta \neq 1$, $\sigma > 0$. Then, following through the method set out by Neyman & Pearson (1933) for the selection of best critical regions, it is easy to show that the appropriate criterion is $M_{.1} = N \ln (\Sigma y^2/N) - \Sigma \ln y^2$, and that if $M_{.1}(\alpha)$ and $M_{.1}(1-\alpha)$ are the α and $1-\alpha$ significance points for $M_{.1}$, then the inequalities

$$M_{.1} > M_{.1}(\alpha) \quad \text{and} \quad M_{.1} < M_{.1}(1-\alpha) \tag{12}$$

define a pair of common best critical regions for testing H_0 against the alternative hypotheses $\delta < 1$ ($\gamma_2 > 0$) and $\delta > 1$ ($\gamma_2 < 0$), respectively, where in each case σ is unknown. Thus in the particular case when there is only one observation per group, $M_{.1}$ is itself the most sensitive criterion possible for testing for a certain type of non-normality. Non-normal distributions of the class considered, which are defined by equation (7), would rarely be expected to approximate to the distribution of actual data, but the above investigation suggests that we may form some idea of the sensitivity to non-normality of $M_{.1}$ for small samples by seeing how good it is as a *test* for non-normality.

* The actual standardized cumulants given above differ slightly in some cases from the theoretical values due to the finite size (10,000) of the populations which Le Roux sampled.

Table 3. *Characteristics of observed distributions of twenty-five values of M_1 for $k = 20$ groups of (a) $n = 20$*, (b) $n = 5$ observations, drawn from a variety of Pearson-type parent distributions*

			Mean			Variance			No. out of 25 exceeding normal theory significance point, P			
n	γ_1^2	γ_2	Expected, normal theory	Found in samples of 25 values of M_1	Expected for $D(s^2)$ III population with same γ_2	Expected, normal theory	Found in samples of 25 values of M_1	Expected $D(s^2)$ III population with same γ_2	$P = 10\%$	$P = 5\%$	$P = 1\%$	$P = 0{\cdot}1\%$
20	0·0	−0·5	19·4	15·0	14·7	39·4	16·5	22·7	0 (0%)	0 (0%)	0 (0%)	0 (0%)
20	0·5	0·7	19·4	25·8	25·9	39·4	92·7	70·8	10 (40%)	5 (20%)	3 (12%)	2 (8%)
20	1·0	0·8	19·4	28·6	26·9	39·4	93·3	76·1	13 (52%)	9 (36%)	4 (16%)	1 (4%)
25	1·2	2·8	19·4	37·4	46·5	39·4	159·0	227·2	21 (84%)	16 (64%)	10 (40%)	5 (20%)
20	0·0	4·1	19·4	39·4	59·1	39·4	329·4	367·1	19 (76%)	17 (68%)	11 (44%)	7 (28%)
5	0·0	−0·5	20·7	19·3	16·2	44·9	36·8	27·8	2 (8%)	1 (4%)	0 (0%)	0 (0%)
5	0·5	0·7	20·7	23·0	27·0	44·9	29·4	76·9	3 (12%)	2 (8%)	0 (0%)	0 (0%)
5	1·0	0·8	20·7	28·0	28·0	44·9	79·0	82·4	13 (52%)	9 (36%)	2 (8%)	0 (0%)
5	1·2	2·8	20·7	26·8	47·8	44·9	67·5	240·0	11 (44%)	8 (32%)	1 (4%)	0 (0%)
5	0·0	4·1	20·7	28·5	61·8	44·9	192·1	401·4	9 (36%)	5 (20%)	1 (4%)	1 (4%)

* In the single case $\gamma_1^2 = 1{\cdot}2$, $\gamma_2 = 2{\cdot}8$ the sample n number taken by Le Roux was 25 instead of 20.

(7·1) *Comparison of $M_{.1}$ with criteria for testing normality*

The usual test criterion for kurtosis is b_2, the sample estimate of the moment coefficient β_2. Geary (1935*a*) proposed as a new test for kurtosis the criterion

$$a = \text{(sample mean deviation)/(sample standard deviation)}.$$

This criterion had the advantage that unlike b_2, the distribution when the null-hypothesis of normality was true could be found fairly readily. Further investigation was carried out by Geary (1935*b*, 1947) to discover whether the power of this test to pick out a departure from normality was comparable to that of b_2. Pearson (1935) studied this question by means of sampling experiments. One such experiment consisted of drawing ten samples each of seventy-six observations from each of three symmetrical populations:

Description	γ_1^*	γ_2^*
Rectangular	0·0	−1·2
Pearson type VII	0·0	4·1
Double exponential	0·0	2·9

* As before the actual standardized cumulants given above differ slightly from the theoretical values for the last two populations due to the finite size (10,000) of the population sampled.

Pearson used his results to show the relative frequency with which the test criteria b_2 and a detected departures from normality and also to show the correlation between the results of the two tests. He published not only his conclusions but also the actual samples which he used and I have used these again to calculate the corresponding values for $M_{.1}$. In my calculation of $M_{.1}$ I have used the population mean (zero in each case), whereas in Pearson's calculations of b_2 and a the sample mean was used. In samples of seventy-six, however, the discrepancy this introduces should not be large. A further difficulty arises due to the fact that the samples are grouped to the nearest unit, resulting in a frequency class having zero deviation from the mean which would give rise to an infinite value of $M_{.1}$. To overcome this I have spaced the values evenly in the interval 0·0–0·5. Thus if there were n values in the zero frequency class these have been assumed to be at $1/(4n)$, $3/(4n)$, ..., $(2n-1)/(4n)$. Fig. 2*a* shows the thirty values of $M_{.1}$ plotted against the corresponding values of b_2 and Fig. 2*b* shows the same values plotted against the values of a. It is seen that in each case marked correlation occurs, and if these graphs are compared with Pearson's Fig. 4 in which he plots a against b_2 it will be seen that the extent of correlation between $M_{.1}$ and b_2 and between $M_{.1}$ and a is similar to that between a and b_2. So far as is known, no tables are available for the significance points of M_1 (or for the equivalent criterion L_1) for the case of seventy-six groups with variance estimates each based on 1 degree of freedom. Also the asymptotic approximations of Bartlett (1937), Hartley (1940) and Box (1949) break down for large k and small numbers of degrees of freedom. Approximate significance points for $M_{.1}$ were therefore calculated by considering the distribution of $L_1^{-1} = \exp\{M_1/k\}$ which ranges from 1 to ∞. General expressions for the moments of L_1 were given by Neyman & Pearson (1931). From these the first four moments of L_1^{-1} were calculated and the appropriate Pearson type curve to approximate the distribution was found to be of type VI. It was assumed therefore that L_1^{-1} was approximately distributed as $cF_{\phi_1\phi_2}$, the curve starting at 1. By equating the first three moments of the two curves the values

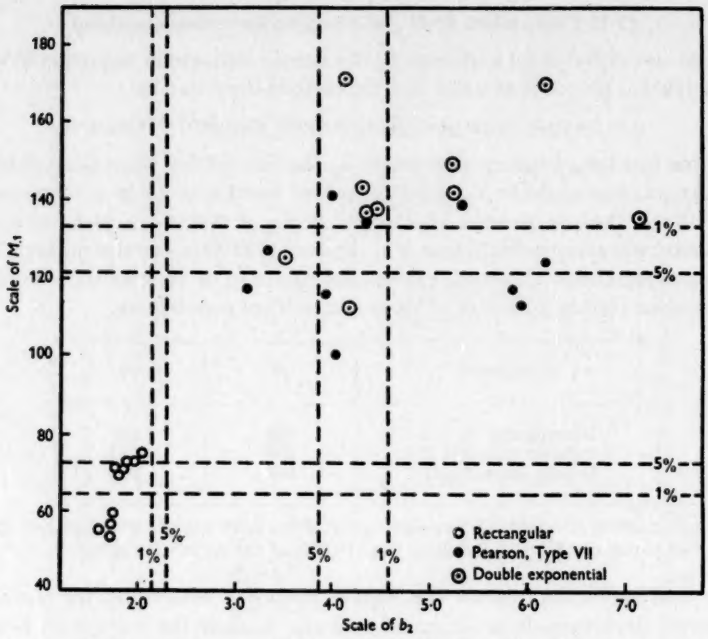

Fig. 2*a*. Comparison of $M_{.1}$ and b_2 in samples of seventy-six.

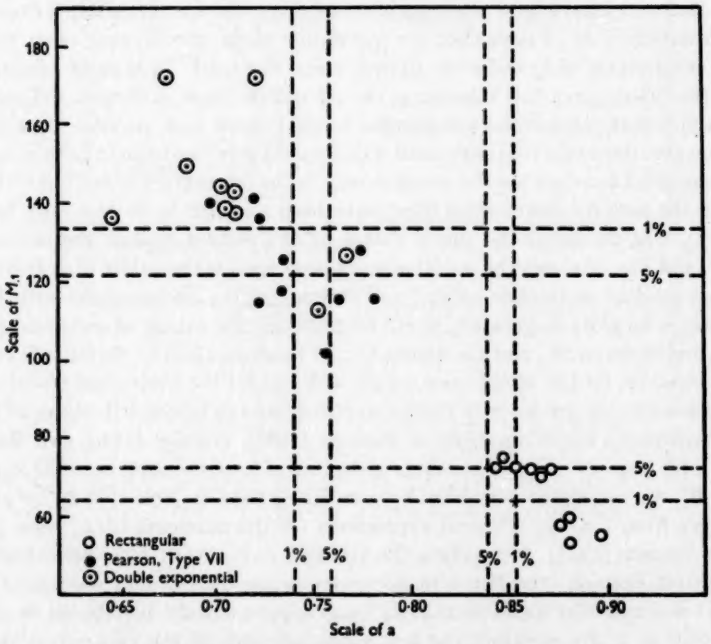

Fig. 2*b*. Comparison of $M_{.1}$ and a in samples of seventy-six.

$c = 2{\cdot}4705$, $\phi_1 = 67{\cdot}1998$, $\phi_2 = 45{\cdot}2904$ were obtained, and the approximate significance points calculated by two-way, four-point harmonic interpolation in the F tables of Merrington & Thompson (1943). The approximate significance points thus found were:

Lower		Upper	
1 %	5 %	5 %	1 %
64·1	72·3	121·2	133·3

Table 4 is an extension of Pearson's Table VII showing the number of values of b_2, a and $M_{.1}$ exceeding their 5 and 1 % significance levels.

Table 4. *Samples of seventy-six; comparison of criteria*

Distribution	Rectangular $\gamma_1^2 = 0{\cdot}0$, $\gamma_2 = -1{\cdot}2$			Pearson type VII $\gamma_1^2 = 0{\cdot}0$, $\gamma_2 = 4{\cdot}1$			Double exponential $\gamma_1^2 = 0{\cdot}0$, $\gamma_2 = 2{\cdot}9$		
	b_2	a	$M_{.1}$	b_2	a	$M_{.1}$	b_2	a	$M_{.1}$
No. of samples:									
within 5 % limits	0	0	3	2	3	5	1	1	1
between 5 and 1 % limits	0	2	3	4	1	2	5	1	1
beyond 1 % limits	10	8	4	4	6	3	4	8	8

For the rectangular and type VII distributions the criterion appears to be less sensitive to kurtosis than b_2 or a; for the double exponential distribution, however, $M_{.1}$ appears to be as sensitive as a and more sensitive than b_2. It is not of course contended that $M_{.1}$ is necessarily a practical test for kurtosis; we are concerned only to show that since b_2 and a are presumably very sensitive criteria for detecting kurtosis, $M_{.1}$ is much more sensitive *even when the sample size is small* than we should desire a test *for homogeneity of variances* to be.

In this connexion it is of some interest to reconsider some data discussed by Bartlett & Kendall (1946). The results appear as a two-way table with fifteen columns and three rows. The entries in the table are the logarithms of variance estimates s^2, each based on about 48 degrees of freedom. The authors show that by taking logarithms the data are brought to a suitable form for the application of analysis of variance. Their analysis is as follows:

	D.F.	S.S.	M.S.
Between rows	2	0·2667	0·1333
Between columns	14	0·1047	0·0075
Residual	28	0·1005	0·0036
Theoretical variance	—	—	0·0020

The first three items are calculated in the usual way, and the theoretical variance is based on the normal theory value $\kappa_2(\ln s^2) \sim 2/(\phi - 1)$. Compared with this theoretical variance the residual mean square is significant ($P = 0{\cdot}01$). It is deduced, therefore, that there is heterogeneity in the residual variance, and methods for further analysing this residual heterogeneity are discussed. Although familiarity with the distribution of their data (of which clearly a large quantity was available) no doubt rendered quite legitimate in this example the assumption by these authors of approximate normality, it is perhaps worth while emphasizing the dependence of this test on that assumption. For in general

$$\kappa_2(\ln s^2) \sim 2/(d\phi - 1),$$

consequently a value for γ_2 of 1·57 would have made the residual and theoretical variances exactly equal. Furthermore, since any value for the theoretical variance greater than 0·0024 would fail to show the residual variance significant at the 5 % level, any hypothesis that the residual variance is homogeneous and γ_2 greater than 0·44 is not contradicted by the data at this level of significance.

8. Other tests of variance homogeneity

The reason for the difference in sensitivity to non-normality between the criterion M_2 for testing homogeneity of means and M_1 for testing homogeneity of variances can be seen as follows. In the analysis of variance (F-form) the criterion for means may be written as P/Q, where $P = \{\Sigma n_i(\bar{y}_i - \bar{y})^2\}/(k-1)$ is the between-groups mean square and $Q = s^2$ is the within-groups mean square. When the null hypothesis is true, P and Q each provides an unbiased estimate of σ^2 whether the observations are drawn from normal populations or not, so that Q is always a standard with which P may be usefully compared. But if in M_1, the criterion for variances, we write $s_i^2 = \sigma^2(1 + x_i)$ and formally expand the logarithms, we obtain $M_1 = \frac{1}{2}\sum_i \phi_i(x_i - \bar{x})^2$ plus terms in higher powers of x_i. For sufficiently large values of ϕ_i we can ignore the higher powers of x_i and we have $M_1/(k-1) \sim p/q$, where

$$p = \{\Sigma\phi_i(s_i^2 - s^2)^2\}/(k-1) \quad \text{and} \quad q = 2\sigma^4.$$

Comparing these with the expressions above we see that asymptotically the M_1 test is like an analysis of variance on the sample variances instead of sample means, *but the quantity p corresponding to the between-groups mean square is compared not with an estimate from the internal evidence of the samples but with a theoretical value of the variance which is appropriate only when the parent distribution is normal.* In general, the variance of a variance estimate s_i^2 is $d^{-1} \times 2\sigma^4/\phi_i$; consequently the appropriate value with which to compare p would be $d^{-1} \times 2\sigma^4$ and not $2\sigma^4$.

From the above discussion it will be seen that to obtain a criterion less dependent on γ_2, information on the variation to be expected in the sample variances gathered from internal evidence in the samples should be utilized. A test of variances should be 'studentized' for the fourth moment just as a test of means is studentized for the second moment.

(8·1) *Other tests not utilizing information within groups*

Other tests of variance homogeneity which do not utilize evidence on variance variability within the samples are equally sensitive to non-normality. In fact for $D(s^2)$ III parent distributions and, asymptotically, for all distributions with finite cumulants any criterion

$f\{s^2_{\phi_1}(\gamma_{21}), s^2_{\phi_2}(\gamma_{22}), \ldots, s^2_{\phi_k}(\gamma_{2k})\}$ will be distributed like $f\{s^2_{d_1\phi_1}(0), s^2_{d_2\phi_2}(0), \ldots, s^2_{d_k\phi_k}(0)\}$. Examples of such tests are those proposed by Cochran (1941) and by Hartley (1950). Cochran's criterion is $g = s^2_{\max.}/\Sigma s^2_i$, whilst that proposed by Hartley is $F_{\max.} = s^2_{\max.}/s^2_{\min.}$, where $s^2_{\max.}$ and $s^2_{\min.}$ are the largest and smallest of the group variances $s^2_1, s^2_2, \ldots, s^2_i, \ldots, s^2_k$. Using the same approximation as that adopted by Hartley it is a simple matter to calculate the true chance of exceeding the normal theory significance points for a $D(s^2)$ III parent population. For instance with k groups of 21 observations drawn at random from such a population in which $\gamma_2 = 1$ the approximate percentage chance of exceeding the normal theory 5 % significance point would be:

	$k = 5$	$k = 10$	$k = 20$
M_1	17·5	25·2	37·3
$F_{\max.}$	17·3	23·3	30·3

The discrepancies using $F_{\max.}$ are seen to be of similar magnitude to those found with M_1, and similar results are to be expected using Cochran's g. The multivariate tests for the constancy of the variance covariance matrix from one group of observations to another (Wilks, 1932; Bishop, 1939; Box, 1949, 1950) would be expected to be equally dependent on the assumption of multivariate normality.

Sensitivity to non-normality is found also with sequential tests. For example, Wald (1947) discussed the use of the sequential likelihood ratio test of the hypothesis H_0 that $\sigma^2 = \sigma^2_0$ when the alternative H_1 is that $\sigma^2 = \sigma^2_1$ and the observations $y_1, y_2, \ldots$ were drawn from normal populations with known mean η. Suppose the observations were drawn, not from a normal universe, but from the double χ population whose distribution is given by equation (7). Then it will be found that the logarithm of the likelihood ratio $L(\gamma_2)$ is equal to $\delta L(0)$ (where $L(0)$ is the logarithm of the likelihood ratio when the parent distribution is normal). Consequently if the quantity $L(0)$ is calculated when γ_2 is not zero and referred to limits $\ln(\beta/(1-\alpha))$ and $\ln((1-\beta)/\alpha)$, this is equivalent to referring the actual likelihood $\delta L(0)$ to the limits $\delta \ln\{\beta/(1-\alpha)\}$ and $\delta \ln\{(1-\beta)/\alpha\}$. The actual risks of error of the two kinds will therefore be α' and β' chosen so that $\beta'/(1-\alpha') = \{\beta/(1-\alpha)\}^\delta$ and $(1-\beta')/\alpha' = \{(1-\beta)/\alpha\}^\delta$. For $100\alpha = 100\beta = 5$ the percentage risk of errors of each of the two kinds is given below for various values of γ_2:

γ_2	-1	0	1	2
$100\alpha' = 100\beta'$	0·28	5·00	12·32	18·66

As with the tests with fixed sample sizes the result will be approximately true for any population with finite cumulants when the average sample sizes are not small. Roughly $\alpha' = \alpha^\delta$, $\beta' = \beta^\delta$ for small values of α and β. Similar difficulties would be expected with other sequential tests relating to the true value of and equality of variances; for example, with the tests proposed by Girshick (1946) and the tests based on ranges suggested by Cox (1949).

(8·2) *A test utilizing within-group information*

A test on variances less sensitive to parent non-normality must clearly utilize the within-group information in some way. Although the author believes that a better approach is available, one immediately practical method is to split up the groups of observations into sub-groups, and carry out an analysis of variance between and within groups on the logarithms of the sub-group variances, following Bartlett & Kendall (1946), who have shown the value of the logarithmic transformation in bringing variance data to a form suitable for the application of analysis of variance. The question of what sizes should best be taken for the subgroups requires further investigation.

Table 5. *Criteria calculated for ten samples of twenty groups of twenty observations drawn from a rectangular population*

Source of sample*		Test of means	Tests of variances			
Page	Rows	M_2	M_1	M_2 on logarithms of subgroup variances		
				2 sub-groups of 10	5 sub-groups of 4	10 sub-groups of 2
I	1–20	14·5	6·7	26·2	10·3	34·0
I	21–40	22·7	8·6	18·6	16·1	18·1
I and II	41–50 1–10	13·8	10·8	29·6	29·3	19·3
II	11–30	12·6	16·7	30·7	35·0	18·1
II	31–50	19·6	9·9	40·6	20·3	10·3
III	1–20	26·9	5·2	17·9	10·8	11·6
III	21–40	12·5	16·1	40·8	29·6	19·8
III and IV	41–50 1–10	18·0	7·4	34·1	21·6	34·1
IV	11–30	23·8	10·3	37·9	28·2	19·5
IV	31–50	11·8	10·3	30·6	32·4	29·6
Mean: Found		17·6 ± 1·7	10·2 ± 1·2	30·7 ± 2·6	23·4 ± 2·8	21·4 ± 2·7
Expected on normal theory		19·5	19·4	27·0	22·1	21·3
Variance: Found		29·3 ± 15·6	13·9 ± 7·2	66·1 ± 35·3	78·7 ± 42·0	71·0 ± 37·9
Expected on normal theory		40·1	39·4	79·0	51·5	48·1

* The groups of observations consisted in each case of the first twenty columns of numbers from pages of Fisher & Yates's tables. The page numbers and the numbers of the rows are shown above.

The results of a small sampling experiment of some interest in this connexion are set out in Table 5. Ten samples of twenty groups of twenty observations were drawn from the table of random numbers prepared by Fisher & Yates (1938). The parent distribution was thus effectively rectangular. As a test of group to group homogeneity of *means* a 'between and within groups' analysis of variance was performed for each of the ten samples of twenty groups and the ten resulting values of M_2 calculated. These are shown in the third column of the table. Four different tests for homogeneity of the variances were applied to each sample. The first was the M_1 test, the results for which are shown in the fourth column of the

table. In the remaining three columns are shown the values of M_2 for analysis of variance performed on the logarithms of subgroup variances. The groups of twenty observations were divided into subgroups in three different ways: (i) two subgroups of ten observations, (ii) five subgroups of four, (iii) ten subgroups of two. The means and variances of the observed values together with their approximate standard errors and the values expected on normal theory are shown at the bottom of the table.

With so few samples, only large discrepancies of course would be detectable. We see that the test on means shows no evidence of departure from the values expected on normal theory even though the parent population is rectangular. The M_1 test for homogeneity of variance on the other hand shows extremely large departures, the mean being only about half the value expected on normal theory (for the rectangular parent distribution the asymptotic mean value is two-fifths of the normal-theory value). In contrast, it is seen that all the tests for homogeneity of variance based on M_2 give values agreeing fairly well with what would be expected if it could be assumed that the distribution of the logarithm of the variance was exactly normal, an assumption far from true, particularly for subgroups with only two observations.

9. Discussion

It has frequently been suggested that a test of homogeneity of variances should be applied before making an analysis of variance test for homogeneity of means in which homogeneity of variance is assumed. The present research suggests than when, as is usual, little is known of the parent distribution, this practice may well lead to more wrong conclusions than if the preliminary test was omitted. It has been shown (Welch, 1937; David & Johnson, 1951*b*; Box, 1952; and Horsnell, 1953) that in the commonly occurring case in which the group sizes are equal, or not very different, the analysis of variance test is affected surprisingly little by variance inequalities. Since this test is also known to be very insensitive to non-normality it would be best to accept the fact that it can be used safely under most practical conditions. To make the preliminary test on variances is rather like putting to sea in a rowing boat to find out whether conditions are sufficiently calm for an ocean liner to leave port!

When the groups of observations were of unequal size and differences in variances might occur it would seem logical to replace the usual analysis of variance criterion which uses a pooled estimate of within-groups variance by the alternative criterion proposed by Welch (1951) and by James (1951), i.e. $\sum_i w_i(\bar{x}_i - \bar{x})^2$, where $w_i = n_i/s_i^2$. This criterion is robust to inequality of variance and almost certainly to non-normality also. (In fact, where inequality of variance might occur it would seem most logical to use Welch's criterion even if the groups were equal.)

When a criterion for testing a statistical hypothesis is derived (for example, by the likelihood ratio method), it is usually necessary for purposes of mathematical convenience to over-simplify the specification of the problem. We should not be surprised therefore if an examination of the resulting criterion shows that the assumptions have sometimes, so to speak, been interpreted rather too literally. For this reason it is most important that derived criteria should be studied for robustness.

The property of robustness I believe to be even more important in practice than that the test should have maximum power and that the statistics employed should be fully efficient.

Where necessary I believe that the latter qualities should be sacrificed to ensure the former.* On the other hand, I do not think that we need necessarily go to the extreme of using non-parametric tests when it may well be that more powerful robust parametric tests can be found.

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* Since writing the above a very interesting paper by J. W. Tukey (1948) has come to my notice which has many points of contact with the present paper and which expresses similar views to the above.

APPENDIX

Details of sampling experiments. Values of M_1 for $k = 20$ variance estimates, based on sets of $n = 20$, 25 and 5 observations randomly drawn from various parent populations

Parent population: Pearson type and values of (γ_1^2, γ_2)	II (0·0, −0·5)		III (0·5, 0·75)		I (1·0, 0·8)		IV (1·2, 2·8)		VII (0·0, 4·1)	
Sample	$n = 20$	$n = 5$	$n = 20$	$n = 5$	$n = 20$	$n = 5$	$n = 25$	$n = 5$	$n = 20$	$n = 5$
1	21·26	13·18	24·70	22·76	21·06	33·06*	50·74***	33·69*	94·48***	25·49
2	18·22	20·75	20·59	33·14*	42·46**	21·75	26·92	36·16*	44·32***	16·47
3	15·80	16·06	30·39?	35·76*	22·85	21·24	42·64**	33·95*	22·18	25·25
4	18·26	13·49	29·23?	18·00	20·70	13·28	80·00***	25·83	47·21***	32·19?
5	10·49	13·31	37·26**	19·34	33·93*	27·84	21·96	25·08	36·78*	14·58
6	12·66	22·95	23·84	18·29	57·62***	32·44?	18·70	32·27?	32·58*	17·41
7	10·87	19·16	16·64	28·49	42·16**	39·57**	35·44*	30·66?	40·81**	34·02*
8	19·83	32·19?	21·68	16·70	19·64	33·59*	42·77**	9·27	29·91?	36·64*
9	14·65	16·76	18·84	17·79	22·57	17·88	29·44?	19·40	24·42	28·46
10	13·63	14·90	19·42	30·27?	33·70*	38·44*	36·11*	15·59	36·09*	32·77*
11	18·43	20·12	16·94	25·75	30·41?	39·10*	35·14*	32·87*	37·86**	32·44?
12	20·27	17·61	27·26?	24·33	36·27*	33·83*	34·18*	18·44	15·45	24·58
13	10·84	14·88	28·72?	23·34	21·43	20·79	42·70**	33·08*	49·13***	25·01
14	15·98	27·92	18·77	17·01	37·74**	26·30	36·25*	28·50	37·89**	32·31?
15	21·22	20·55	31·54*	21·82	17·31	35·89*	52·25***	20·86	37·84**	21·01
16	12·53	14·30	19·39	22·08	31·55*	20·83	50·98***	27·67	24·68	17·64
17	8·35	17·80	30·62?	12·40	16·00	14·03	42·23**	35·38*	24·92	19·03
18	9·03	18·68	24·97	23·11	20·36	16·03	28·58?	42·13**	36·73*	27·42
19	16·55	17·26	19·97	29·34	20·30	30·89?	34·33*	26·62	72·17***	31·93?
20	10·07	18·12	13·09	25·70	21·16	46·49**	37·14**	28·14	57·89***	23·77
21	16·29	14·66	44·74***	24·12	28·55?	33·38*	45·22***	32·37?	31·22*	27·95
22	10·72	14·26	55·37***	25·27	31·44*	32·70?	30·32?	17·93	28·72?	87·95***
23	11·69	36·22*	32·86*	21·66	30·69?	30·22?	24·36	16·63	12·91	25·87
24	15·39	28·80	15·32	20·73	27·57	21·75	27·81?	15·16	71·40***	34·10*
25	21·61	21·05	22·30	19·05	28·70	19·75	29·25?	34·18*	36·56*	19·96

Significance points for M_1 (Bartlett's approx.):		10%	5%	1%	0·1%
	$n = 25$	27·60	30·58	36·72	44·46
	$n = 20$	27·70	30·70	36·86	44·63
	$n = 5$	29·58	32·78	39·36	47·65
Significance at these levels denoted by		?	*	**	***

APPROXIMATING TO THE DISTRIBUTIONS OF MEASURES OF DISPERSION BY A POWER OF χ^2

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1. Introduction

The use of range in place of sample variance in various statistical tests has been the subject of a number of recent papers. In order to make range and means of sets of ranges amenable to certain lines of attack, two approximations have been proposed for the case of a normal parent population. The first consists in replacing range by a multiple of χ with an appropriate number of degrees of freedom. The two parameters at choice allow the mean and variance of range to be matched by those of the approximating form. This method has been discussed by Patnaik (1950), Hartley (1950) and Florin (1950); David (1951) gives tables of the necessary constants. The second approach, due to Cox (1949), replaces range by a χ^2 form. Pearson (1952) has compared the errors arising from the two methods. He points out that either can be regarded as an approximate transformation to the χ^2 form, and thus a number of standard tests are available. For instance, Bartlett's test for homogeneity of variance or the short-cut tables prepared by David (1952) may be used. Similarly, the ratio of a pair of ranges can be replaced by an approximate F-ratio.

Below we graduate the distribution of range by a multiple of a suitable power of χ^2. The extra parameter allows the first three moments to be matched, and much smaller errors result. By inverting the relationship an approximate transformation to the χ^2 form results. It should be noted that, while the χ approximation is dimensionally correct, the others are not. Thus the increased accuracy is obtained at the price of some loss of applicability of the approximation.

The method gives good results up to sample size 20 and the necessary constants are tabled. Results for the mean of a set of m ranges are also satisfactory and constants are given for m from 2 to 5 and n from 2 to 10. As there are no probability integral tables available for this statistic, these constants may be used to supply approximate percentage points from those of χ^2.

As the power of the statistic needed depends on sample size, the use of Bartlett's test or of an F-ratio test is restricted to values for samples of the same size. In this case David's 5 and 1 % short-cut tables can be used. These have been transformed into equivalent tables for the ratio of maximum range to minimum range. Enough values have been checked by quadratures to ensure that these latter tables are sufficiently accurate for most purposes.

In a paper to be published shortly, moment constants for the first quasi-range are given. This statistic is defined as the difference between the greatest-but-one and the least-but-one of the ordered values in the sample. For sample sizes beyond 17 it is more efficient than range as an estimator of standard deviation. It is also less influenced by the presence of an occasional 'rogue' observation, or by departures from the normal model. The same method of graduation gives good results and a table of constants is given.

Mean deviation can be graduated in the same way, and transformation constants and 5 and 1 % short-cut tables are provided.

Wilson & Hilferty (1931) have shown that, for large degrees of freedom, the cube root of χ^2 is almost normal. This leads to an approximate normalization of statistics that can be graduated by a power of χ^2. An example is given in the section on range.

Another transformation of some interest is the use of the logarithm of χ^2. This statistic has a variance independent of the parent value of the standard deviation, and can be used if it is desired to apply analysis of variance techniques to a set of observed standard deviations. However, while the statistic approaches normality in large samples, it does so rather slowly. Bartlett & Kendall (1946) suggest that the method should be used with caution for sample sizes below 10. We see that the logarithmic transformation when applied to range, or any of the other statistics considered below, will have the same properties. The variance of $\ln w$ will still be independent of the parent standard deviation and will be approximately

$$\alpha^2\left[\frac{d^2}{dx^2}\ln\Gamma(x)\right]_{x=\frac{1}{2}\nu}.$$

The errors introduced by the approximation to w will be small in comparison with the departures from normality of $\ln\chi^2$. Thus conditions for the use of the logarithmic transformation of range or mean deviation will be similar to those for its use with standard deviation.

It seems likely that other applications of this method of graduation may occur. Thus non-central χ^2 can be well approximated by a central χ^2 on a suitably chosen number of degrees of freedom. Still better results could be obtained by the present method. The use of frequency curves of the form

$$a+b(\chi^2)^\alpha \quad (\chi^2 \text{ on } \nu \text{ d.f.})$$

might also repay investigation.

2. Determination of constants

Consider the quantity

$$x = (\tfrac{1}{2}\chi^2)^\alpha,$$

where χ^2 is on ν degrees of freedom. The moments of x are readily found; they are given by the following relation:

$$\mu'_r = \frac{\Gamma(r\alpha+\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu)}. \tag{1}$$

Using Stirling's series, the expansions below follow at once:

$$V = \frac{\sqrt{\mu_2}}{\mu'_1} = \alpha\sqrt{\left(\frac{2}{\nu}\right)}\left\{1+\frac{(\alpha-1)^2}{2\nu}+\frac{(\alpha-1)^2(5\alpha-1)(\alpha-5)}{24\nu^2}+\ldots\right\}, \tag{2}$$

$$\frac{\sqrt{\beta_1}}{V} = \frac{\mu'_1\mu_3}{\mu_2^2} = 3-\frac{1}{\alpha}+\frac{2(\alpha-1)^2}{\nu}+\ldots. \tag{3}$$

On solving (2) and (3) for ν we obtain

$$\nu = \frac{2}{(3V-\sqrt{\beta_1})^2}\left\{1-\frac{(2V-\sqrt{\beta_1})^2(V+\sqrt{\beta_1})}{2(3V-\sqrt{\beta_1})}+\ldots\right\}. \tag{4}$$

Inversion of (2) gives the series

$$\alpha = \alpha_0-\frac{\alpha_0(\alpha_0-1)^2}{2\nu}+\frac{\alpha_0(\alpha_0-1)^2(13\alpha_0^2+2\alpha_0+1)}{24\nu^2}+\ldots, \tag{5}$$

where

$$\alpha_0 = \sqrt{\left(\frac{\nu}{2}\right)}V.$$

If the values of V and β_1 for a particular statistic u be inserted in (4) and (5), we shall have the results

$$u \text{ is approximately distributed like } (\chi^2/c)^\alpha, \tag{6}$$

$$cu^\lambda \text{ is approximately distributed like } \chi^2. \tag{7}$$

The constant c is chosen to make the means of the two variates in (6) agree. We have

$$\alpha = \frac{1}{\lambda} \quad \text{and} \quad \log c = \log 2 + \lambda\{\log \Gamma(\alpha + \tfrac{1}{2}\nu) - \log \Gamma(\tfrac{1}{2}\nu) - \log \bar{u}\}. \tag{8}$$

The expression for $\log c$ is readily evaluated using the log factorial tables prepared by Brownlee (1923).

For small ν or large α, (4) and (5) will not be sufficiently accurate. The following procedure was found to be satisfactory for range, quasi-range and mean deviation.

Since (5) comes directly from (2), the use of an approximate value for ν, while affecting the β_1 match of u and x does not affect the matching of mean and variance. Thus, for $\nu > 10$ it is found sufficient to round to the nearest integer and substitute this value in the first two terms of (5) to determine α.

When $10 > \nu > 4$ it is better to take ν to the nearest 0·1, and three terms of (5) are needed. In practice it was found preferable to solve

$$V^2 + 1 = \frac{\Gamma(2\alpha + \tfrac{1}{2}\nu)\,\Gamma(\tfrac{1}{2}\nu)}{\{\Gamma(\alpha + \tfrac{1}{2}\nu)\}^2}. \tag{9}$$

Thus, for mean deviation with $n = 5$, two terms of (5) give $\alpha = 0{\cdot}554$. Using (9) twice we find

$$\alpha = 0{\cdot}55, \quad V^2 = 0{\cdot}13655,$$

$$\alpha = 0{\cdot}56, \quad V^2 = 0{\cdot}14131.$$

Inverse linear interpolation for the required value of V^2 (0·13929) gives $\alpha = 0{\cdot}5558$. The use of trial values of α with a spacing of 0·01 obviates interpolation in Brownlee's tables and gives a maximum error in α of a few units in the fifth place.

When $\nu < 4$ the series (4) still gives very good results. Thus for range with $n = 3$ it gives 2·08, the true value being 2·05. However, it is now advisable to retain the second decimal in ν. Thus it becomes necessary to solve (9) and the relation

$$\sqrt{\beta_1}\, V^3 + 3V^2 + 1 = \frac{\Gamma(3\alpha + \tfrac{1}{2}\nu)\,\{\Gamma(\tfrac{1}{2}\nu)\}^2}{\{\Gamma(\alpha + \tfrac{1}{2}\nu)\}^3}. \tag{10}$$

This is readily done using trial values of ν rounded to 0·02 to obviate interpolation in the log factorial tables. Thus, for range with $n = 4$, (4) gives $\nu = 3{\cdot}23$. The values of V^2 and $\sqrt{\beta_1}$ at each of the four points

$$\nu = 3{\cdot}20, \quad \alpha = 0{\cdot}52, \qquad \nu = 3{\cdot}24, \quad \alpha = 0{\cdot}52,$$

$$\nu = 3{\cdot}20, \quad \alpha = 0{\cdot}53, \qquad \nu = 3{\cdot}24, \quad \alpha = 0{\cdot}53,$$

are then found using (9) and (10). Double inverse linear interpolation for the required values of V^2 and $\sqrt{\beta_1}$ then gives

$$\nu = 3{\cdot}202, \quad \alpha = 0{\cdot}5257.$$

With an interval of 0·04 in trial values of ν, neglect of second differences only affects α slightly in the fifth decimal place in the most unfavourable case.

It might be desirable to use a fixed value of λ for different sample sizes. This value could be chosen as a compromise between the values given for the various sample sizes in Table 1. The values of ν for each sample size will be found from

$$\nu = \nu_0 + (\alpha - 1)^2 - \frac{(\alpha^2 - 1)^2}{3\nu_0} + \ldots, \tag{11}$$

where $$\nu_0 = \frac{2\alpha^2}{V^2} \quad \text{and} \quad \alpha = \frac{1}{\lambda}.$$

When $\alpha = \frac{1}{2}$ this reduces to the series developed by Florin (1950) for the χ approximation. Two decimals in ν are usually adequate, and for $\nu_0 < 4$ we solve (9) for ν using trial values suggested by (11).

It sometimes happens that all but one or two samples are of the same size; the λ appropriate to this size may then be used throughout. For the exceptional samples ν and c are found from (11) and (8); for the standard size the values of Table 1 are used.*

3. RANGE

In order to apply an approximate test of homogeneity to a set of ranges, each for the same sample size, we first transform to approximate χ^2 values using the constants of Table 1. Then the usual Bartlett test is applied to these values. The adequacy of the approximate transformation to the χ^2 form was investigated as follows.

Tabular values of u, as near as possible to the percentage points listed below, were transformed by (7). The approximate probability integral was then found from tables of the χ^2 integral. This was compared with the exact value in tables of the probability integral of range prepared by Pearson & Hartley (1942). Errors in the approximate probability integral, expressed in units of the fourth decimal, are tabled below:

% point / n	0·5	5	20	50	80	95	99·5
3	−1	−2	+2	+3	−2	−2	−1
5	−1	−4	+5	+5	−3	−2	0
10	−3	−3	+5	+4	−4	−1	0
15	−2	0	+7	0	−3	0	+1
20	−2	+1	+5	−2	−3	+1	0

David's (1952) 5 and 1 % short-cut tables for the maximum F-ratio, transformed by (7), gave Tables 3 and 4. Exact values of ν, not the rounded values of Table 1, were taken. A four-point Lagrange formula was used when interpolating for ν, and harmonic values of ν were employed when it exceeded 10. For small ν the use of variance ratio became impossible, but the ratio of $s_{\text{min.}}$ to $s_{\text{max.}}$ was quite satisfactory for interpolation. The table for the 1 % values was not taken beyond $n = 10$ because interpolation and rounding errors begin to affect probabilities appreciably beyond this value.

We shall now apply a check to these values. If, in a set of k ranges, M denotes the ratio of maximum range to minimum range, we have

$$\Pr.(M \geqslant M_0) = 1 - k\int_0^\infty \{F(M_0 x) - F(x)\}^{k-1} f(x)\, dx. \tag{12}$$

* All numbered Tables have been placed together at the end of the paper.

Using values of M_0 in Tables 3 and 4, the following probabilities were found by quadrature:

% point		$k=2$	$k=6$	$k=11$
5 %	$n=5$	0·0506	0·0514	0·0526
	$n=10$	0·0508	0·0504	0·0516
	$n=20$	0·0515	0·0516	0·0496
1 %	$n=5$	0·0104	0·0106	0·0107
	$n=10$	0·0103	0·0102	0·0094

These values show a slight bias, but are satisfactory for most purposes. In view of the size of the error at $n=5$, $k=11$, the true probability was also found at the nominal 5 % level for $n=4$, $k=11$; it was 0·0513. While the effect on probability of a change of $+0{\cdot}01$ in the ratio varies over the tables, average values are $-0{\cdot}0004$ for the 5 % tables and $-0{\cdot}0002$ for the 1 % table.

We now turn to another application of the method. The Wilson-Hilferty approximation uses the fact that, for large ν,

$$\mathscr{E}\left(\frac{\chi^2}{\nu}\right)^{\frac{1}{3}} = 1-\frac{2}{9\nu}+O(\nu^{-3}), \quad \text{var.}\left(\frac{\chi^2}{\nu}\right)^{\frac{1}{3}} = \frac{2}{9\nu}+O(\nu^{-3}).$$

In addition, the distribution of this quantity is nearly normal. Applying this fact to range we find

$w_5^{0\cdot6099}$ is approximately Nr.$\{1{\cdot}6443, 0{\cdot}3844\}$,*

$w_{10}^{0\cdot4881}$ is approximately Nr.$\{1{\cdot}7163, 0{\cdot}2197\}$.

The errors in the approximate probability integral, found from the normal integral, are shown below in units of the fourth decimal:

% point	0·5	5	20	50	80	95	99·5
$n=5$	+20	+19	−15	−16	−10	+4	−1
$n=10$	+ 1	− 1	− 3	− 1	+ 2	0	−1

While the approximation is good at $n=10$, there are large errors when n is as small as 5.

4. Mean range

One method used when estimating dispersion in large samples consists in splitting the sample into subgroups. The mean of the subgroup ranges is then used as an estimator of standard deviation. Grubbs & Weaver (1947) have discussed the optimum method of splitting the sample. The mean of m ranges, each for the same sample size, will often arise naturally in other ways. Thus we might have m independent samples in which the variate has a fixed dispersion, but may take different mean levels in the various samples.

Table 2 gives the constants needed to transform the mean of m ranges, each for a sample of n, to the χ^2 form by means of (7). Except in one case, discussed later, no simple method is available for checking this approximation directly. Consequently the following procedure was used.

* Nr.$\{\mu, \sigma\}$ is used to denote a normal distribution, having mean $=\mu$ and standard deviation $=\sigma$.

While the two variates in (6) agree in their first three moments, the β_2 values will differ. Using (1) the following values were found for this β_2 difference. The β_2 of u was the larger of the two throughout:

n	$m=1$	$m=2$	$m=3$	$m=4$
2	0	−0·013	−0·012	−0·008
3	−0·005	−0·004	−0·002	−0·002
4	−0·010	−0·005	−0·003	−0·002
10	−0·015	−0·009	−0·006	−0·004

The percentage point tables prepared by Pearson & Merrington (1951) show that, for the region of the (β_1, β_2) plane concerned, a given β_2 error has a greater effect for large values of β_1 than for small; its effect varies but little with β_2. For the largest value of β_1 in the above table ($n = 2, m = 1$) the transformation is exact. The next largest, at $m = n = 2$, also shows the second largest β_2 difference.

The probability density function for a single range is of simple form when $n = 2$, and we find by integration that the mean of two such ranges has the density:

$$\frac{8}{\sqrt{(2\pi)}} e^{-\frac{1}{4}\bar{w}^2} \int_0^{\bar{w}} \frac{e^{-\frac{1}{4}x^2}}{\sqrt{(2\pi)}} dx.$$

The probability integral of $\bar{w}$ was found by quadrature, and the previous method of checking used to give the following errors in units of the fourth decimal:

% point	0·5	5	20	50	80	95	99·5
Error	−3	−5	+4	+11	−1	−4	+1

Since (β_1, β_2) approaches the normal point as m increases, this test, together with those made for $m = 1$ in the previous section, may be assumed to cover the least favourable cases.

5. THE FIRST QUASI-RANGE

In samples too large for the direct use of range, it may be necessary to randomize the observations before splitting them into groups. For instance, we may suspect that values are not independent of the order in which they are recorded. The lack of uniqueness may also be a drawback to the use of average range in some circumstances.

In such cases the first quasi-range, defined as the difference between the greatest-but-one and the least-but-one of the ordered values in the sample, may prove useful. Using the tables of expected values of range given by Tippett (1925) and reproduced in *Tables for Statisticians and Biometricians*. Vol. 2, the mean value of this statistic can be found from

$$\begin{aligned}\mathscr{E}(x_{n-1} - x_2) = 2\mathscr{E}(x_{n-1}) &= 2n(n-1)\int_{-\infty}^{\infty} F(1-F)^{n-2} dF \\ &= 2n(n-1)\int_{-\infty}^{\infty} (1-F)^{n-2} dF - 2n(n-1)\int_{-\infty}^{\infty} (1-F)^{n-1} dF \\ &= n\mathscr{E}(w_{n-1}) - (n-1)\mathscr{E}(w_n).\end{aligned}$$

Using the first three moments of quasi-range* the constants of Table 1 were found. These enable approximate percentage points to be deduced from those of χ^2 by using (6). Errors in the approximate probability integral of quasi-range found from (7) are given below in units of the fourth decimal:

n \ %point	0·5	5	20	50	80	95	99·5
10	−1	−2	+1	+4	0	−1	0
20	−2	0	+6	+2	−4	−1	+1
30	−2	+1	+4	−2	−2	−1	+1

In order to avoid double interpolation in the Hartley-Pearson (1950) tables of χ^2, values of ν have been rounded to the nearest even integer when above 30.

6. Mean deviation

Work with this statistic was on the same lines as for range. Errors in the probability integral appear below; these are expressed in units of the fourth decimal:

n \ %point	0·5	5	20	50	80	95	99·5
3	−1	−2	+2	+ 3	−2	−2	−1
4	−2	−5	+7	+10	−6	−3	+1
5	−2	−4	+8	+ 3	−6	−1	+1
6	−2	−3	+5	+ 3	−3	0	+1
10	−1	−1	+1	+ 3	−1	−1	0

As no tables are available beyond sample size 10, β_2 differences in (6) were again examined.

Series expansions for β_1 and β_2, due to Geary, are given by Pearson in an Editorial Note to a paper by Godwin & Hartley (1945). Godwin (1948) gives a minor correction to the β_2 series. Fisher (1920) derived the exact distribution of mean deviation when $n = 4$, and gave the values $\beta_1 = 0{\cdot}297$, $\beta_2 = 3{\cdot}28$. The series gives the values 0·299 and 3·252 respectively.

An examination of Fisher's paper shows that the fourth moment of σ_1 (the mean deviation estimator of σ) given there should be replaced by

$$\frac{9\pi}{512}(5\pi - 7\alpha) + \frac{165\pi}{384\sqrt{2}}, \quad (\alpha = \cos^{-1}\tfrac{1}{3}).$$

Using this value we obtain $\beta_1 = 0{\cdot}2983$, $\beta_2 = 3{\cdot}252$, in very good agreement with Geary's series.

The values of β_2 differences found are shown below:

n	3	4	5	6	10	15	20
β_2 difference	−0·005	−0·027	−0·017	−0·011	−0·008	−0·004	−0·003

* To be published shortly in a paper accepted for the *Annals of Mathematical Statistics*.

The large values at 4 and 5 are reflected in the errors found in the approximate probability integral. As mean deviation moves steadily towards normality with increase of n, we can expect errors beyond 10 to be small.

Tables 5 and 6 give the approximate upper 5 and 1 % values of the ratio of maximum mean deviation to minimum mean deviation in a set of k values. Using (12) the following values of probability were found from these tables by quadrature:

% point		$k = 2$	$k = 6$	$k = 11$
5 %	$n = 4$	0·0510	0·0520	0·0522
	$n = 10$	0·0506	0·0506	0·0509
1 %	$n = 4$	0·0104	0·0108	0·0111
	$n = 10$	0·0100	0·0101	0·0099

It will be seen that errors are similar to those for range. For sample sizes below 10 it will be found that the critical ratio differs very little for range, mean deviation or standard deviation.

The use of rounded values of ν in Table 1 obscures the fact that λ tends to a limit as n increases; this can readily be proved by using Geary's series. This suggests that, with a slight loss of accuracy, a fixed value of λ could be employed. It turns out that 1·8 is a satisfactory value, and appropriate tables are being prepared. The use of a fixed λ greatly increases the flexibility of the method by removing the restriction to a fixed sample size.

I should like to thank Mr D. F. Mills for his assistance in the preparation of the tables. I am indebted to Prof. Pearson for his advice on the presentation of this work, and for making an advance copy of the short-cut tables of critical variance ratios available.

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Table 1. *Transformation constants for range, mean deviation and first quasi-range*

n	Range			Mean deviation			n	First quasi-range		
	ν	λ	log c	ν	λ	log c		ν	λ	log c
							10	8·0	1·7289	0·3590
							11	9·3	1·7144	0·3884
2	1·00	2	Ī·6990	1·00	2	0·3010	12	11	1·6683	0·4441
3	2·05	1·9619	Ī·7632	2·05	1·9619	0·5766	13	12	1·6810	0·4477
4	3·20	1·9029	Ī·8453	3·35	1·8437	0·7630	14	14	1·6271	0·5110
5	4·5	1·8298	Ī·9393	4·6	1·7992	0·8851	15	16	1·5845	0·5648
6	6·0	1·7489	0·0412	5·9	1·7655	0·9823	16	18	1·5497	0·6117
7	7·7	1·6700	0·1426	7·2	1·7437	1·0612	17	20	1·5202	0·6532
8	9·5	1·6046	0·2315	8·4	1·7394	1·1235	18	22	1·4950	0·6908
9	12	1·5078	0·3543	9·6	1·7355	1·1778	19	24	1·4730	0·7250
10	14	1·4643	0·4215	11	1·7164	1·2326	20	27	1·4257	0·7857
11	16	1·4292	0·4794	12	1·7310	1·2695	21	29	1·4102	0·8129
12	19	1·3605	0·5748	13	1·7428	1·3036	22	32	1·3738	0·8629
13	22	1·3075	0·6537	15	1·6918	1·3596	23	36	1·3236	0·9290
14	26	1·2392	0·7521	16	1·7036	1·3874	24	38	1·3153	0·9482
15	30	1·1860	0·8338	17	1·7144	1·4136	25	42	1·2755	1·0036
16	34	1·1427	0·9036	18	1·7235	1·4383	26	46	1·2413	1·0530
17	40	1·0783	1·0030	20	1·6872	1·4799	27	50	1·2115	1·0977
18	46	1·0276	1·0858	21	1·6966	1·5012	28	54	1·1853	1·1384
19	52	0·9865	1·1569	22	1·7053	1·5215	29	58	1·1618	1·1758
20	60	0·9358	1·2431	23	1·7129	1·5409	30	62	1·1409	1·2104

If the statistic is u, then we have:

cu^{λ} has approximately the χ^2 distribution on ν degrees of freedom.

Table 2. *Transformation constants for the mean of m ranges, each for a sample of size n*

n	$m = 2$			$m = 3$		
	ν	λ	log c	ν	λ	log c
2	2·22	1·8376	0·1644	3·46	1·7822	0·3918
3	4·4	1·8539	0·1749	6·7	1·8282	0·3791
4	6·7	1·8339	0·2217	10	1·8302	0·4064
5	9·3	1·7822	0·2948	14	1·7740	0·4823
6	12	1·7385	0·3617	18	1·7337	0·5449
7	15	1·6849	0·4364	23	1·6633	0·6355
8	19	1·5995	0·5434	29	1·5838	0·7372
9	23	1·5375	0·6283	34	1·5477	0·7952
10	28	1·4620	0·7285	42	1·4613	0·9066
n	$m = 4$			$m = 5$		
	ν	λ	log c	ν	λ	log c
2	4·7	1·7562	0·5408	6·0	1·7310	0·6572
3	9·0	1·8152	0·5178	11	1·8339	0·6044
4	14	1·7816	0·5739	17	1·8054	0·6529
5	19	1·7547	0·6257	24	1·7446	0·7330
6	24	1·7319	0·6731	30	1·7310	0·7719
7	30	1·6812	0·7449	38	1·6683	0·8544
8	38	1·5959	0·8503	48	1·5870	0·9567
9	46	1·5354	0·9336	58	1·5288	1·0381
10	56	1·4605	1·0327	70	1·4607	1·1300

If the statistic is u, then we have:

cu^{λ} has approximately the χ^2 distribution on ν degrees of freedom.

Table 3. *Approximate upper 5 % points of the ratio of maximum value to minimum value in a set of k independent ranges. Each range is for a sample of size n from a normal population of fixed standard deviation*

n \ k	2	3	4	5	6	7	8	9	10	11	12
3	6·28	9·32	11·9	14·2	16·3	18·2	20·0	21·7	23·3	24·8	26·3
4	3·96	5·31	6·32	7·20	7·95	8·63	9·24	9·76	10·3	10·7	11·1
5	3·15	4·02	4·63	5·10	5·53	5·93	6·26	6·55	6·80	7·05	7·28
6	2·74	3·37	3·82	4·16	4·47	4·71	4·93	5·14	5·32	5·50	5·66
7	2·49	2·99	3·34	3·61	3·85	4·04	4·22	4·37	4·51	4·63	4·75
8	2·32	2·75	3·04	3·27	3·46	3·62	3·75	3·88	3·99	4·09	4·18
9	2·20	2·58	2·83	3·03	3·19	3·32	3·44	3·55	3·64	3·73	3·81
10	2·11	2·45	2·68	2·84	2·99	3·11	3·21	3·31	3·39	3·46	3·53
12	1·97	2·26	2·46	2·60	2·72	2·82	2·90	2·98	3·05	3·11	3·16
15	1·85	2·09	2·25	2·37	2·46	2·54	2·61	2·67	2·73	2·78	2·83
20	1·72	1·92	2·05	2·14	2·21	2·28	2·34	2·38	2·42	2·46	2·50

Table 4. *Approximate upper* 1 % *points for the range ratio*

n \ k	2	3	4	5	6	7	8	9	10	11	12
3	14·1	21·0	26·7	31·8	36·4	40·7	44·8	48·5	52·1	55·4	58·7
4	6·91	9·28	11·0	12·4	13·7	14·8	15·8	16·7	17·6	18·5	19·1
5	4·87	6·16	7·06	7·76	8·44	9·04	9·51	9·93	10·5	10·8	11·2
6	3·96	4·80	5·40	5·86	6·30	6·66	6·99	7·25	7·51	7·76	7·96
7	3·44	4·08	4·52	4·86	5·16	5·43	5·66	5·86	6·06	6·26	6·46
8	3·11	3·63	3·98	4·27	4·50	4·71	4·88	5·05	5·18	5·31	5·44
9	2·88	3·33	3·62	3·86	4·05	4·20	4·34	4·47	4·60	4·71	4·81
10	2·71	3·10	3·35	3·56	3·73	3·87	4·00	4·10	4·20	4·29	4·38

Table 5. *Approximate upper* 5 % *points for the ratio of maximum mean deviation to minimum mean deviation in a set of k independent values. Each value is derived from a sample of n from a normal population of fixed standard deviation*

n \ k	2	3	4	5	6	7	8	9	10	11	12
3	6·28	9·32	11·9	14·2	16·3	18·2	20·0	21·7	23·3	24·8	26·3
4	3·96	5·32	6·31	7·20	7·95	8·64	9·22	9·74	10·3	10·8	11·2
5	3·17	4·04	4·64	5·13	5·55	5·96	6·30	6·59	6·85	7·09	7·31
6	2·74	3·37	3·81	4·16	4·47	4·70	4·93	5·14	5·32	5·49	5·65
7	2·48	2·98	3·33	3·60	3·82	4·02	4·19	4·34	4·48	4·62	4·74
8	2·30	2·72	3·01	3·23	3·42	3·57	3·72	3·84	3·95	4·04	4·13
9	2·17	2·54	2·78	2·97	3·13	3·26	3·37	3·48	3·57	3·65	3·73
10	2·07	2·39	2·61	2·77	2·91	3·02	3·12	3·21	3·29	3·36	3·43
12	1·92	2·19	2·36	2·50	2·60	2·70	2·77	2·84	2·91	2·96	3·01
15	1·77	1·99	2·13	2·24	2·32	2·39	2·45	2·51	2·56	2·60	2·64
20	1·63	1·80	1·91	1·99	2·05	2·10	2·15	2·19	2·22	2·26	2·28
30	1·49	1·61	1·68	1·74	1·78	1·82	1·85	1·88	1·91	1·93	1·94
60	1·31	1·38	1·43	1·46	1·48	1·50	1·52	1·54	1·55	1·56	1·57
∞	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00

Table 6. *Approximate upper* 1 % *points for the mean deviation ratio*

n \ k	2	3	4	5	6	7	8	9	10	11	12
3	14·1	21·0	26·7	31·8	36·4	40·7	44·8	48·5	52·1	55·4	58·7
4	6·92	9·30	11·0	12·3	13·6	14·8	15·9	16·8	17·6	18·4	19·0
5	4·89	6·18	7·12	7·83	8·49	9·04	9·50	10·0	10·5	10·9	11·2
6	3·96	4·79	5·41	5·86	6·30	6·66	6·98	7·26	7·52	7·75	7·96
7	3·43	4·07	4·51	4·85	5·15	5·41	5·64	5·85	6·06	6·26	6·44
8	3·08	3·60	3·95	4·22	4·45	4·67	4·84	5·00	5·15	5·28	5·40
9	2·83	3·27	3·56	3·80	3·98	4·15	4·29	4·42	4·53	4·63	4·72
10	2·65	3·02	3·27	3·47	3·64	3·77	3·89	3·99	4·09	4·18	4·26

THE POWER FUNCTION OF SOME TESTS BASED ON RANGE

By H. A. DAVID

1. Introduction

In a number of recent papers (Patnaik, 1950; Hartley, 1950*a*; David, 1951) tests based on the sample range have been developed to deal with the analysis of simple and double classifications as well as of some more complex orthogonal designs. It is our purpose to compare the power function of these short-cut tests with that of the analysis of variance procedure usually employed.

The new test criteria which may be used in place of the F-ratio are of two forms, depending on whether the test is one for a main factor or an interaction. In the former case the test function is the ratio of a single range to an independent mean range estimator of the population standard deviation σ, all ranges being taken over a *small* number of variates. In this situation a mean range can be closely approximated by a χ-distribution (with fractional degrees of freedom ν) so that the test ratio may be referred to tables of the 'Studentized' range q. The same χ-approximation allows us also to deal with the ratio of two independent mean ranges which arises in a test for an interaction.

The power of these tests has not been determined for any of the above designs, except in special cases. However, there has been evidence suggesting that the tests are likely to be reasonably powerful. Thus: (*a*) It is known that for a small sample size n the range provides a fairly efficient estimator of σ and that for large n ($n > 12$) a very good estimator of σ is given by the mean of ranges taken over small subgroups of the original sample (Davies & Pearson, 1934; Grubbs & Weaver, 1947). (*b*) For all of the above classifications tables are available giving the 'equivalent degrees of freedom' ν of estimates of error found from mean ranges (see, for example, David, 1951, pp. 408–9). ν is generally not much smaller than the number of error degrees of freedom in the analysis of variance. (*c*) Extensive investigations by quadrature methods made by Lord (1950) show the power of his u-test (Lord, 1947), employing a range estimate of variance, to be little less (on the systematic model) than that of the corresponding t-test, especially when mean ranges are employed.

These considerations suggest that the use of a *mean* range instead of a root mean square as a 'Studentizing' statistic leads to a definite but small loss in power. This will be confirmed in the sequel. It will also be shown that if a random probability model is relevant, range methods are somewhat less powerful than the standard procedure. For the systematic case attention is confined to a number of special alternate hypotheses. The q-test, whilst always retaining the property of unbiasedness (in the Neyman-Pearson sense) of the F-test, is found in certain rather special but important practical situations to be more powerful than the F-test.

2. The random model

We consider first the random set-up

$$x_{ij} = K + u_i + z_{ij} \quad (i = 1, \ldots, l; j = 1, \ldots, m), \tag{1}$$

where K is a constant and the u_i, z_{ij} are all mutually independent normal deviates with variances σ'^2 (for u) and σ^2 (for z). The presence of the u-terms in (1) makes the variance of

the means $\bar{x}_i$ equal to $(\sigma^2 + m\sigma'^2)/m$ instead of σ^2/m; hence the usual ratio of the between-groups mean square to the within-groups mean square is distributed not as F but as $(1+m\zeta^2)F$, where $\zeta = \sigma'/\sigma$, the degrees of freedom of F being $\nu_1 = l-1$, $\nu_2 = l(m-1)$. In this case the single-tailed F-test is known to be the uniformly most powerful test of the null hypothesis that $\zeta = 0$ against alternatives for which $\zeta > 0$.

When range methods are employed, we test the null hypothesis by referring the ratio

$$q_u = \frac{\sqrt{m}\,\text{range}\,(\bar{x}_i)}{\bar{w}/c} \tag{2}$$

to tables of percentage points of the 'Studentized' range q (Pearson & Hartley, 1943; May, 1952; Hartley, 1953). Since the presence of the u-terms multiplies the standard deviation of $\sqrt{m}\,\bar{x}_i$ by the factor $\sqrt{(1+m\zeta^2)}$, q_u will be distributed approximately as $\sqrt{(1+m\zeta^2)}\,q$, where q is the 'Studentized' range for sample size l and degrees of freedom ν.

Table 1. *Power of the q-test for a simple classification (random model); power of corresponding F-test* $= 1-\beta$

1*a*. $\alpha = 0{\cdot}05$, $1-\beta = 0{\cdot}90$

l \ m	3	6	∞
4	0·90	0·90	0·90
6	0·86	0·88	0·89
8	0·85	0·87	0·87
10	0·83	0·86	0·86

1*b*. $\alpha = 0{\cdot}01$, $1-\beta = 0{\cdot}90$

l \ m	3	6	∞
4	0·89	0·89	0·89
6	0·87	0·88	0·88
8	0·85	0·86	0·87
10	0·83	0·85	0·85

1*c*. $\alpha = 0{\cdot}05$, $1-\beta = 0{\cdot}75$

l \ m	3	6	∞
4	0·74	0·74	0·74
6	0·68	0·72	0·73
8	0·67	0·70	0·71
10	0·65	0·69	0·69

1*d*. $\alpha = 0{\cdot}01$, $1-\beta = 0{\cdot}75$

l \ m	3	6	∞
4	0·73	0·74	0·74
6	0·70	0·71	0·72
8	0·68	0·69	0·69
10	0·64	0·67	0·67

As pointed out by Patnaik (1950) the power function of the q-test can be obtained from tables of the probability integral of q and may be compared with the power of the corresponding F-test above. We do this by employing a general method for the comparison of the power function of two different tests of the same hypothesis (see Hartley, 1950*b*). For an assigned significance level α we find the value of ζ for which the power of the F-test has a selected value $1-\beta$; we then find the power of the q-test for the same value of ζ. If $F(\alpha; \nu_1, \nu_2)$ is the upper 100α % point of F for degrees of freedom ν_1, ν_2, it follows that the value of ζ is given by

$$1+m\zeta^2 = F(\alpha; \nu_1, \nu_2)/F(1-\beta; \nu_1, \nu_2).$$

Hence, using a similar notation for probability points of q, the power of the q-test for the same value of ζ is

$$\Pr.[\sqrt{(1+m\zeta^2)}\,q > q(\alpha; l, \nu)] = \Pr.[q > q(\alpha; l, \nu)\sqrt{\{F(1-\beta; \nu_1, \nu_2)/F(\alpha; \nu_1, \nu_2)\}}].$$

Results for the cases $\alpha = 0{\cdot}05,\ 0{\cdot}01$; $\beta = 0{\cdot}10,\ 0{\cdot}25$ and selected values of l and m are given in Table 1. Thus from Table 1 *a* we see, for example, that in testing for the significance of differences in means in eight categories, each containing six observations ($l = 8$, $m = 6$), a power of 0·90 given by the standard test is reduced to 0·87 when the present q-test is employed. It is evident from the table as a whole that the loss of power is small, especially in the commonest situations where the number of groups l is not very large nor the number within groups m very small. The effect of l on the power is clearly more marked than that of m. As $m \to \infty$ the problem reduces to that of comparing the range of the group means with the between-groups mean square as criteria for detecting departure of the between-groups population variance from $\sigma'^2 = 0$.

2·1. *A modified form of the test*

It is of some interest to determine the loss in power when only the denominator of the F-ratio is replaced by a range estimator (since presumably most of the reduction in power is due to the use of a single range in the numerator of q). The resulting test criterion is

$$m\Sigma(\bar{x}_i - \bar{x})^2/(\overline{w}/c)^2,$$

which is distributed approximately as F with degrees of freedom $l-1, \nu$. Using a significance level α, its power to detect a σ'^2 which the ordinary F-ratio test returns as significant with power $1-\beta$, is given approximately by the Incomplete Beta-ratio

$$\left.\begin{aligned} &I_t(\tfrac{1}{2}l(m-1), \tfrac{1}{2}(l-1)) = I_t(\tfrac{1}{2}\nu_1, \tfrac{1}{2}\nu_2), \\ \text{where} \qquad &t = \nu_2/(\nu_2 + \nu_1 F'), \quad F' = F(\alpha; \nu_1, \nu)\, F(1-\beta; \nu_1, \nu_2)/F(\alpha; \nu_1, \nu_2). \end{aligned}\right\} \tag{3}$$

We have confined ourselves to the case $m = 3$ for which the drop in power due to the use of the 'Studentized' range criterion is most substantial. (To two-decimal accuracy the results are the same for $l = 6, 8, 10$.)

$l = 4$

β \ α	0·05	0·01
0·10	0·90	0·90
0·25	0·74	0·74

$l = 6,\ 8,\ 10$

β \ α	0·05	0·01
0·10	0·89	0·89
0·25	0·74	0·73

As expected, the loss in power is small, and there may be circumstances in which the present hybrid test criterion is of practical value.

2·2. *Test for a double classification*

Very similar results to those in Table 1 are obtained for the double classification

$$x_{ij} = K + u_i + v_j + z_{ij} \quad (i = 1, \ldots, l;\ j = 1, \ldots, m).$$

These are set out in Table 2, which is relevant to the power of the 'Studentized' range test for the existence of the u_i (the power of tests for the v_i follows by symmetry). m has been taken as 4, 7, ∞ so as to give F-ratios with the same degrees of freedom as corresponding entries in Table 1 with $m = 3, 6, \infty$.

2·3. *Power of modified F-ratio test for interaction*

In testing for an interaction a criterion of the form $F_1 = \left(\dfrac{\overline{w}_1/c_1}{\overline{w}_2/c_2}\right)^2$ arises through the use of range methods. F_1 is distributed approximately as an F-ratio with modified degrees of

freedom ν_1', ν_2' which, as well as c_1, c_2, are obtainable from appropriate tables given by David (1951). It is clear that a power function comparison of the F_1- and F-tests may be made precisely as in (3). We have taken two cases typical of a split-plot experiment, but

Table 2. *Power of the q-test for a double classification into l treatments and m blocks (random model)*

2*a*. $\alpha = 0{\cdot}05$, $1-\beta = 0{\cdot}90$

l \ m	4	7	∞
4	0·90	0·89	0·90
6	0·88	0·89	0·89
8	0·87	0·87	0·87

2*b*. $\alpha = 0{\cdot}01$, $1-\beta = 0{\cdot}90$

l \ m	4	7	∞
4	0·90	0·89	0·89
6	0·89	0·88	0·88
8	0·87	0·87	0·87

2*c*. $\alpha = 0{\cdot}05$, $1-\beta = 0{\cdot}75$

l \ m	4	7	∞
4	0·75	0·74	0·74
6	0·71	0·72	0·73
8	0·71	0·71	0·71

2*d*. $\alpha = 0{\cdot}01$, $1-\beta = 0{\cdot}75$

l \ m	4	7	∞
4	0·75	0·74	0·74
6	0·73	0·71	0·72
8	0·71	0·70	0·69

similar results will hold in general. We consider the comparison of 'error (1)' against 'error (2)', which is often carried out as a preliminary test of significance:

$l = 5$, $m = 4$, $n = 3$

β \ α	0·01	0·05
0·10	0·87	0·88
0·25	0·71	0·72

$l = 3$, $m = 3$, $n = 5$

β \ α	0·01	0·05
0·10	0·87	0·87
0·25	0·72	0·73

3. The systematic set-up

We confine ourselves to the simple classification

$$x_{ij} = K + A_i + z_{ij} \quad (i = 1, \ldots, l;\ j = 1, \ldots, m),$$

where the systematic effects A_i are subject to the condition $\Sigma A_i = 0$. Similar results will be seen to hold for other models.

The case of two groups ($l = 2$) need not be considered here as the test ratio (2) is simply

$$\sqrt{m}\,|\,\bar{x}_1 - \bar{x}_2\,|/(\bar{w}/c),$$

and is equivalent to Lord's u-criterion appropriate to a two-sided test for the significance of the difference of two means (see § 1).

For $l > 2$ no simple procedure for the evaluation of the power of the q-test seems possible. We shall proceed by illustrating that, as in § 2·1, the loss in power due to the mean-range denominator alone is small, and by then investigating the more important changes in power introduced by the single-range numerator.

From Tang's (1938) tables it is easy to obtain the reduction in power when the degrees of freedom in the denominator of the F-ratio are reduced from $l(m-1)$ to ν, to give what we may term the F_R-test. We give two examples in the manner of Tang's table with

$$\phi = \sqrt{(m\Sigma A_i^2/l)}.$$

Example. (i) $l = 6$, $m = 6$; (ii) $l = 8$, $m = 3$.

Values of $1-\beta$

$\alpha = 0{\cdot}05$

ϕ	(i)		(ii)	
	F	F_R	F	F_R
1·5	0·738	0·728	0·711	0·696
2·0	0·952	0·948	0·942	0·937

$\alpha = 0{\cdot}01$

ϕ	(i)		(ii)	
	F	F_R	F	F_R
2·0	0·824	0·810	0·762	0·736
2·5	0·974	0·969	0·952	0·941

3·1. *The power of the range test*

A consideration of the numerator of q alone is clearly equivalent to an investigation of an ordinary range test, i.e. one for which σ is known. (2) now reduces to

$$\sqrt{m}\,\text{range}\,(\bar{x}_i)/\sigma,$$

a non-central range if the expectations of the $\bar{x}_i$ differ. We shall write, using a dash to denote non-centrality,

$$w' = \text{range}\,(y_i),$$

where the y_i follow normal distributions with expectations α_i (where $\Sigma\alpha_i = 0$) and unit variances.

Writing $z(x) = e^{-\frac{1}{2}x^2}/\sqrt{(2\pi)}$, the probability integral of w' is given by

$$P_l(w') = \sum_{i=1}^{l}\int_{-\infty}^{\infty} z(y_i-\alpha_i)\prod_{h\neq i}^{l}\left(\int_{y_i}^{y_i+w'} z(y_h-\alpha_h)\,dy_h\right)dy_i. \tag{4}$$

The power $(1-\beta)$ of the range test at significance level α, namely, $1-P_l(w_\alpha)$, is a symmetric function of the l expectations α_i.

In view of the constraint $\Sigma\alpha_i = 0$, the power is also a symmetric function of any $l-1$ of the α_i but does not appear to be expressible (unlike the corresponding probability integral of non-central χ^2 which depends only on $\Sigma\alpha_i^2$) as a function of less than these $l-1$ independent parameters (compare § 4).

We therefore confine attention to a few important special cases for which the evaluation of $P_l(w_\alpha)$ by approximate quadrature is practicable.

(*a*) *The case of a single outlying mean.* For this model we may write

$$\alpha_j = -\mu/l \quad (j = 1,\ldots,l-1), \qquad \alpha_l = (l-1)\mu/l.$$

From (4) we have after a simple translation

$$P_l(w_\alpha) = (l-1)\int_{-\infty}^{\infty} z(x)\left(\int_x^{x+w_\alpha} z(\xi)\,d\xi\right)^{l-2}\left(\int_x^{x+w_\alpha} z(\eta-\mu)\,d\eta\right)dx + \int_{-\infty}^{\infty} z(x-\mu)\left(\int_x^{x+w_\alpha} z(\xi)\,d\xi\right)^{l-1}dx.$$

The second term is very small, as it represents the contribution to $P_l(w_\alpha)$ when the outlier gives rise to the smallest observation. The power of the corresponding χ^2-test was obtained

from the tables of non-central χ^2 given by Patnaik (1949) and Fix (1949) and from the $\nu = \infty$ line of Tang's tables, with $\lambda = \Sigma\alpha_i^2 = (l-1)\mu^2/l$ and $\phi = \sqrt{(\lambda/l)}$. In a few cases Patnaik's χ^2-approximation to χ'^2 had to be used.

Table 3. *Comparison of χ^2- and w-tests for a single outlying mean. Values of the power $(1-\beta)$*

α	ϕ \ l	2	6		10	
		χ^2, w	χ^2	w	χ^2	w
0·05	3	0·563	0·53	0·56	—	—
	4	0·807	0·82	0·83	0·76	0·80
	5	0·942	0·96	0·97	0·94	0·96
	6	0·989	—	—	0·994	0·996
0·01	4	0·600	0·62	0·63	0·54	0·61
	5	0·831	0·88	0·88	0·82	0·87
	6	0·952	0·98	0·98	0·97	0·98

It will be seen that except for $l = 2$, when the two tests are equivalent, the range test is slightly more powerful than the χ^2-test. This is not surprising in a test for outliers and a similar conclusion holds when, for example, there are two outliers, one at each end.*

(*b*) *The case of two equal clusters.* For simplicity take l even, then this set-up is defined by

$$\alpha_i = -\tfrac{1}{2}\mu \quad (i = 1, \ldots, \tfrac{1}{2}l),$$
$$= \tfrac{1}{2}\mu \quad (i = \tfrac{1}{2}l+1, \ldots, l),$$

and $P_l(w_\alpha)$ given by

$$P_l(w_\alpha) = \tfrac{1}{2}l\int_{-\infty}^{\infty} z(x)\left(\int_x^{x+w_\alpha} z(\xi)\,d\xi\right)^{\frac{1}{2}l-1}\left(\int_x^{x+w_\alpha} z(\eta-\mu)\,d\eta\right)^{\frac{1}{2}l} dx$$
$$+ \tfrac{1}{2}l\int_{-\infty}^{\infty} z(x-\mu)\left(\int_x^{x+w_\alpha} z(\xi)\,d\xi\right)^{\frac{1}{2}l}\left(\int_x^{x+w_\alpha} z(\eta-\mu)\,d\eta\right)^{\frac{1}{2}l-1} dx.$$

A procedure similar to that in (*a*) yields the following comparisons:

$\alpha = 0{\cdot}05$

ϕ \ l	2	6		10	
	χ^2, w	χ^2	w	χ^2	w
3	0·563	0·823	0·74	0·94	0·82
4	0·807	0·981	0·95	0·999	0·98

For $l = 2$, this model is identical with (*a*). In other cases the range method is less powerful.

* Results reached by Dixon (1950) are of interest in this context. He plots for various tests for outliers, including the above *w*- and χ^2-tests, 'performance' curves obtained by sampling methods and finds the range test superior for all μ. However, his 'performance' is not measured in quite the same way as our 'power', as he discards all samples for which $\bar{x}_l$ is not the largest value in the sample.

4. SOME GENERAL COMMENTS

We now consider to what extent the conclusions regarding the range test of § 3·1 can be converted into conclusions about 'Studentized' range tests. Case (*a*) is of interest in suggesting that when the models there discussed are relevant a better test than the standard one of the analysis of variance can be obtained by replacing the F-ratio by

$$\sqrt{m}\,\text{range}\,(\bar{x}_i)/s,$$

where

$$s = \sqrt{\{\Sigma\Sigma(x_{ij}-\bar{x}_i)^2/l(m-1)\}},$$

giving a 'Studentized' range test with root-mean-square denominator. The test criterion range $(\bar{x}_i)/(\overline{w}/c)$ may also lead to a more powerful test, especially when the number of degrees of freedom in the numerator is large, but the issue is somewhat obscured by the small loss in power due to the use of $\overline{w}/c$ in place of s. This does not, of course, contradict certain optimum properties of the analysis of variance tests proved by Hsu (1941) and Wald (1942). However, it shows that the non-central range distribution does not depend on $\Sigma\alpha_i^2$ alone, as otherwise Hsu's result would not permit it to lead to a more powerful test than the F-ratio.

These comments provide no rigorous conclusions. It would be possible, following lines similar to those of Hartley (1948), to obtain exact upper bounds for the difference between the power functions of the F- and q-tests in terms of the maximum difference between the power functions of the χ and range tests. But as the distribution of non-central range is not available, this approach cannot at present lead us to any numerical gauge of accuracy.

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SOME SIMPLE APPROXIMATE TESTS FOR POISSON VARIATES

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1. Introduction

If events, such as accidents or stoppages of a machine, occur randomly in time at a true rate λ, the number of events in a fixed time t follows a Poisson distribution of mean λt. In inverse sampling, with n fixed, the time t up to the nth event is distributed as $(2\lambda)^{-1}\chi^2_{2n}$, where χ^2_{2n} denotes a χ^2 variate with $2n$ degrees of freedom. Barnard (1946) has pointed out that the latter result leads to convenient 'sequential' tests of hypotheses about λ. For example, if we have inverse samples (n_1, t_1) and (n_2, t_2) from two populations, the hypothesis that $\lambda_1 = \lambda_2$ is tested by referring $F = t_1 n_2/t_2 n_1$ to the variance-ratio tables with $(2n_1, 2n_2)$ degrees of freedom. In the present note we show that, with a slight modification, Barnard's tests apply almost exactly to direct Poisson sampling in which t is fixed and n is a random variable. We obtain, in particular, a convenient test for the equality of Poisson means, although the main use of the method is likely to be in more complicated situations where λ is the product of unknown parameters.

2. The basic approximation

In direct Poisson sampling in which the number of events occurring in a fixed time is recorded, we have

$$\text{prob (no. of events} \geqslant n) = \sum_{r=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^r}{r!} = \text{prob}\left(\frac{1}{2\lambda}\chi^2_{2n} < t\right), \tag{1}$$

and

$$\text{prob (no. of events} \geqslant n+1) = \text{prob}\left(\frac{1}{2\lambda}\chi^2_{2n+2} < t\right). \tag{2}$$

If we wish to make an approximation to prob (no. of events $> n$) in which the number of events is treated as a continuous variate, it is reasonable to take a quantity intermediate between (1) and (2). A natural choice is

$$\text{prob (no. of events} > n) \simeq \text{prob}\left(\frac{1}{2\lambda}\chi^2_{2n+1} < t\right),$$

i.e. we calculate probabilities as if

$$2\lambda t \text{ is distributed as } \chi^2_{2n+1}. \tag{3}$$

Thus the suggestion is that if we observe n events in a fixed time t, we may work approximately as if n is fixed and $2\lambda t$ is distributed as χ^2_{2n+1}. The remainder of the paper is concerned with the consequences of (3).

3. Comparison of two populations

If we sample two populations with rates of occurrence λ_1, λ_2 and in times t_1, t_2 observe n_1, n_2 events, then, according to (3),

$$\frac{t_1(n_2+\frac{1}{2})\lambda_1}{t_2(n_1+\frac{1}{2})\lambda_2}. \tag{4}$$

is distributed approximately as F with $(2n_1+1, 2n_2+1)$ degrees of freedom. Thus we may test the hypothesis that $\lambda_1 = \lambda_2$ by referring

$$F = \frac{t_1(n_2+\frac{1}{2})}{t_2(n_1+\frac{1}{2})} \tag{5}$$

to the F tables with $(2n_1+1, 2n_2+1)$ degrees of freedom. Also $(100-2\alpha)\,\%$ confidence intervals for λ_1/λ_2 may be obtained from

$$\frac{t_2(n_1+\frac{1}{2})}{t_1(n_2+\frac{1}{2})}F_- < \frac{\lambda_1}{\lambda_2} < \frac{t_2(n_1+\frac{1}{2})}{t_1(n_2+\frac{1}{2})}F_+, \tag{6}$$

where F_-, F_+ are the lower and upper $\alpha\,\%$ points of F with $(2n_1+1, 2n_2+1)$ degrees of freedom.

Example. Observations of the spinning of one batch of wool gave 5 ends down in 200 spindle hours and of a second batch 12 ends down in 180 spindle hours.

Assuming that the occurrence of ends down is random for each batch, we may test for the difference between batches by

$$F = \frac{200}{180} \times \frac{12{\cdot}5}{5{\cdot}5} = 2{\cdot}53,$$

with (11, 25) degrees of freedom. The 5 % point in the ordinary F tables is 2·20, the $2\frac{1}{2}$ % point 2·56. Thus in a two-sided test the difference between batches is very nearly significant at 5 %. The lower and upper $2\frac{1}{2}$ % points of F with (11, 25) degrees of freedom are 1/3·16 and 2·56, so that by (6) a 95 % confidence interval for the ratio of the stoppage rates is

$$\frac{180\times 5{\cdot}5}{200\times 12{\cdot}5}\times\frac{1}{3{\cdot}16} < \frac{\lambda_1}{\lambda_2} < \frac{180\times 5{\cdot}5}{200\times 12{\cdot}5}\times 2{\cdot}56,$$

i.e.

$$0{\cdot}125 < \lambda_1/\lambda_2 < 1{\cdot}01.$$

The test of the difference between batches could equally well be done by the conventional χ^2 method, but the provision of a simple confidence interval for the ratio of the stoppage rates is a useful additional feature of the present approach.*

4. Accuracy of the test

The test (5) may be expected to be accurate for large samples; its accuracy for small samples may be investigated as follows. The distribution of n_1, n_2 corresponding to given λ_1, λ_2, t_1 and t_2 is exactly

$$\text{prob}\,(n_1, n_2) = e^{-\lambda_1 t_1}\frac{(\lambda_1 t_1)^{n_1}}{n_1!}\,e^{-\lambda_2 t_2}\frac{(\lambda_2 t_2)^{n_2}}{n_2!}. \tag{7}$$

If we first determine the critical region of the test (5), we can then find the exact probability associated with the test by adding (7) over all points in the critical region. Table 1 gives the results of such calculations. In all cases a one-sided test of the hypothesis $\lambda_2 = \lambda_1$ against alternatives $\lambda_2 > \lambda_1$ has been examined.

The general conclusion from Table 1 is that except when the population means are very small, the approximate F test gives the probability of errors of the first kind sufficiently accurately for practical purposes. For samples of the same size, the test may be considered satisfactory at the 0·05 level if the true mean exceeds one, and satisfactory at the 0·01 level if the true mean exceeds two.

* *Note added in proof.* The derivation of confidence intervals for the ratio of Poisson means has recently been considered in detail by Chapman (1952).

It is natural to compare the approximate F test with the exact test of Przyborowski & Wilenski (1940), which is based on the distribution of n_1, n_2 conditional on n_1+n_2. The comparison is difficult because the latter test is discrete and the true size of the critical region at, say, the 5 % level is appreciably less than 5 %, when small numbers are involved. The critical region of the approximate F test appears always to consist of the critical region of the exact test with some additional points. It is thus roughly equivalent to Barnard's C.S.M. test (Barnard, 1947).

Table 1. *Exact probabilities associated with various nominal significance levels of the approximate F test of the hypothesis* $\lambda_2 = \lambda_1$ *against alternatives* $\lambda_2 > \lambda_1$

(*a*) *Samples of equal size,* $t_1 = t_2$

Population mean, $\lambda_1 t_1 = \lambda_2 t_2$	Nominal significance level			
	0·1	0·05	0·01	0·001
1	—	0·031	0·001	—
1·5	—	0·049	0·004	—
2	0·124	0·059	0·007	0·0001
3	—	0·065	0·011	—
4	—	0·062	0·012	—
5	—	0·058	0·012	—
6	0·101	0·054	0·012	0·0013

(*b*) *First sample size the larger,* $t_1 = 3t_2$

Smaller population mean, λt_2	Nominal significance level	
	0·05	0·01
1	0·048	0·008
1·5	0·055	0·009
2	0·055	0·010
3	0·056	0·011
4	0·056	0·010
5	0·051	0·010

(*c*) *Second sample size the larger,* $t_2 = 3t_1$

Smaller population mean, λt_1	Nominal significance level	
	0·05	0·01
1	0·012	3×10^{-5}
1·5	0·038	5×10^{-4}
2	0·056	0·002
3	0·059	0·010
4	0·059	0·013
5	0·057	0·012

The approximate F test is an interesting consequence of (3) and may sometimes be preferred to conventional approximate χ^2 methods. The detailed discussion does, moreover, show that (3) may lead to accurate results even in very small samples; in § 7 we shall consider an application where a great simplification is achieved by the use of (3).

5. Confidence intervals for a single mean

The approximation (3) may be used to test the hypothesis that $\lambda = \lambda_0$, or to obtain confidence intervals for λ. An interval of confidence coefficient $(100-2\alpha)\,\%$ is thus given approximately by

$$\tfrac{1}{2}\chi^2_{2n+1,-} < \lambda t < \tfrac{1}{2}\chi^2_{2n+1,+}, \tag{8}$$

where $\chi^2_{\nu,\pm}$ are the upper and lower $\alpha\,\%$ points of χ^2 with ν degrees of freedom. This may be compared with Garwood's (1936) confidence interval, which in the present notation is

$$\tfrac{1}{2}\chi^2_{2n,-} < \lambda t < \tfrac{1}{2}\chi^2_{2n+2,+}. \tag{9}$$

The interval (9) has a confidence coefficient of *at least* $(100-2\alpha)\,\%$ whatever the true mean λt; the true confidence coefficient is a serrated function of λt which, for small λt, has most of its values appreciably greater than $(100-2\alpha)\,\%$.

The interval (8) is always narrower than (9), and the true confidence coefficient is sometimes less than $(100-2\alpha)\,\%$. In many practical applications it would be reasonable to assume that over a number of applications of the method the true means λt are distributed randomly with respect to the serrations of the graph of the confidence coefficient. In this case it would be justifiable to use a confidence interval for which the average confidence coefficient over any fairly small range of λt is at least $(100-2\alpha)\,\%$. The interval (8) appears to satisfy this condition provided that n and α are not very small, and it might therefore be claimed that (8) is preferable to (9). The point is, however, of little practical importance because the reduction in width from (9) to (8) is only small.

Example. Seven stoppages of a machine are observed in a certain period. 90 % confidence intervals for the population number of stoppages are, according to (8), (3·63, 12·50) and according to (9), (3·29, 13·15).

6. Relation with inverse sampling

The close connexion between the tests given in §§ 3 and 5 and those based on inverse sampling has been mentioned briefly in §§ 1 and 2. This connexion will now be discussed in more detail.*

Tests based on the measurement of the intervals between randomly occurring events have been described in full by Maguire, Pearson & Wynn (1952). In particular, their test for the significance of the difference between the rates of occurrence in two series is as follows. Let n_1, n_2 be fixed and let t_1, t_2 be random variables defined as the times in the two series up to the n_1, n_2th events. Then $F = t_1 n_2/t_2 n_1$ may be tested *exactly* as a variance ratio with $(2n_1, 2n_2)$ degrees of freedom. The test of § 3, on the other hand, is that if t_1, t_2 are fixed times and if random variables n_1, n_2 are defined as the numbers of events occurring in times t_1, t_2, then $F = t_1(n_2+\frac{1}{2})/t_2(n_1+\frac{1}{2})$ may be tested *approximately* as a variance ratio with $(2n_1+1, 2n_2+1)$ degrees of freedom.

* I am very grateful to Prof. E. S. Pearson for some helpful comments leading to the addition of this section.

In the first case the interval t_1 ends with the n_1th event, but in the second case this is not in general so. The extra degrees of freedom, and the $\frac{1}{2}$'s in the definition of F, may be thought of as accounting for the periods between the last events and the close of the periods of observation. In many applications in which t_1, t_2 are fixed, the precise instants at which the n_1, n_2th events occur would not be known.

It will frequently happen in applications that neither t_1, t_2 nor n_1, n_2 are fixed in advance of the observations. If, however, the periods t_1, t_2 are determined by some random process that is quite uninfluenced by the numbers of events occurring, then conditionally on t_1, t_2, the quantities n_1, n_2 follow Poisson distributions and the approximate F test may be applied. Similarly, if the time intervals are measured up to the n_1, n_2th events, where n_1, n_2 are determined by a random process independent of the observations, then the exact F test with $(2n_1, 2n_2)$ degrees of freedom is applicabl

There are many other possible ways in which t_1, t_2 might be determined; for example, t_1, t_2 might be chosen by some random process correlated with, but not completely determined by n_1, n_2. In such cases it is not possible to find the properties of the above tests without special investigation, although a reasonable general rule is to use the first F test, with $(2n_1, 2n_2)$ degrees of freedom, whenever the intervals t_1, t_2 are ended by events, and the second test otherwise. This rule breaks down in extreme cases; for example, if t_1, t_2 are determined by a likelihood-ratio sequential test for comparing the rates of occurrence, it would clearly be entirely inappropriate to apply either F test.

7. The logarithmic transformation

The basic approximation (3) is very conveniently expressed in logarithmic form, and so may be expected to be useful whenever λ is the product of unknown parameters. If f_n is any function of n we may rewrite (3)

$$\log\frac{f_n}{t} = \log\lambda + \log f_n - \log\tfrac{1}{2}\chi^2_{2n+1}.$$

The $\log\chi^2$ distribution has been studied in detail by Bartlett & Kendall (1946) and by Wishart (1947), who have, in particular, shown that

$$E\log\tfrac{1}{2}\chi^2_\nu = \psi(\tfrac{1}{2}\nu), \quad \operatorname{var}\log\tfrac{1}{2}\chi^2_\nu = \psi'(\tfrac{1}{2}\nu),$$

where ψ, ψ' are the digamma and trigamma functions. It is convenient to choose f_n so that $\log f_n/t$ is an unbiased estimate of $\log\lambda$. Thus we take $f_n = \exp\{\psi(n+\frac{1}{2})\}$, and for the present purpose it is accurate enough to write

$$f_0 = 0{\cdot}14, \quad f_n = n \quad (n = 1, 2, \ldots). \tag{10}$$

Also
$$\operatorname{var}\log\frac{f_n}{t} = \psi'(n+\tfrac{1}{2}) \simeq \left.\begin{array}{ll} 4{\cdot}93 & (n=0), \\ 1/n & (n=1,2,\ldots). \end{array}\right\} \tag{11}$$

Thus if we define a transformed variate by

$$\left.\begin{aligned} z &= \log_{10}\frac{0{\cdot}14}{t} \quad \text{if} \quad n = 0, \\ &= \log_{10}\frac{n}{t} \quad \text{if} \quad n \neq 0, \end{aligned}\right\} \tag{12}$$

then approximately $E(z) = \log_{10}\lambda$ and $\operatorname{var} z = (\log_{10} e)^2 v_i$,

where
$$\left.\begin{aligned} v_i &= 4{\cdot}93 \quad (n=0), \\ &= 1/n \quad (n \neq 0). \end{aligned}\right\} \tag{13}$$

One use of the transformation will now be described.

Suppose observations of stoppage rates are made for k machines on each of which stoppages occur at random, the rates for the different machines being different. Suppose that a change in the process is introduced and that fresh observations are then made. In some cases it would be reasonable to expect a constant proportional change in stoppage rate, i.e. to expect that if the initial stoppage rates are $\lambda_1, \ldots, \lambda_k$ the final stoppage rates will be $\rho\lambda_1, \ldots, \rho\lambda_k$. If the observations are, in the initial period $(n_1, t_1), \ldots, (n_k, t_k)$, and in the final period $(n'_1, t'_1), \ldots, (n'_k, t'_k)$, we may define transformed variates $z_1, \ldots, z_k$ and $z'_1, \ldots, z'_k$ as in (12). Then $u_i = z'_i - z_i$ has mean $\mu = \log_{10}\rho$ and known variance $(\log_{10} e)^2(v_i + v'_i) = (\log_{10} e)^2 w_i$ say. The best estimate of μ is thus

$$\hat{\mu} = \frac{\Sigma u_i w_i^{-1}}{\Sigma w_i^{-1}}, \quad \operatorname{var}\hat{\mu} = \frac{(\log_{10} e)^2}{\Sigma w_i^{-1}}. \tag{14}$$

An almost unbiased estimate of the parameter ρ is

$$\hat{\rho} = \left(1 - \frac{1}{2\Sigma w_i^{-1}}\right) 10^{\hat{\mu}}. \tag{15}$$

An approximate test of the hypothesis of a proportional change in stoppage rate may be obtained by calculating

$$\chi^2_{k-1} = \{\Sigma w_i^{-1}(u_i - \hat{\mu})^2\}(\log_{10} e)^2. \tag{16}$$

Example. The following data were the results of a sampling experiment with $\rho = 2$, the values of T_i, T'_i being chosen arbitrarily.

i	Initial		Final		u_i	$1/w_i$
	T_i (hr.)	n_i	T'_i (hr.)	n'_i		
1	30	2	28	5	0·428	1·429
2	26	4	41	21	0·522	3·360
3	21	2	24	3	0·118	1·200
4	45	18	16	11	0·235	6·828
5	30	4	32	12	0·449	3·000

$$\Sigma w_i^{-1} = 15{\cdot}817, \quad \Sigma u_i w_i^{-1} = 5{\cdot}4611, \quad \Sigma u_i^2 w_i^{-1} = 2{\cdot}1774.$$

The last two columns give

$$u_i = z'_i - z_i = \log_{10} T_i n'_i / T'_i n_i \quad \text{and} \quad w_i^{-1} = (v_i + v'_i)^{-1} = n_i n'_i/(n_i + n'_i),$$

no frequency being zero.

$$\hat{\mu} = \Sigma u_i w_i^{-1}/\Sigma w_i^{-1} = 0{\cdot}345 \quad \text{and} \quad \operatorname{var}\hat{\mu} = (0{\cdot}4343)^2/15{\cdot}817,$$

so that standard error $(\hat{\mu}) = 0{\cdot}109$. There is thus good agreement with the true value $\mu = \log_{10} 2 = 0{\cdot}3010$. ρ is estimated by

$$\hat{\rho} = \left\{1 - \frac{1}{2 \times 15{\cdot}817}\right\} 10^{\hat{\mu}}$$
$$= 2{\cdot}145.$$

The hypothesis of a proportional change in stoppage rate is tested by

$$\chi_4^2 = (2{\cdot}3026)^2 \Sigma w_i^{-1}(u_i - \hat{\mu})^2 = (2{\cdot}3026)^2 \{\Sigma w_i^{-1} u_i^2 - (\Sigma w_i^{-1})\hat{\mu}^2\}$$
$$= 1{\cdot}55, \text{ indicating a good agreement.}$$

This problem could be tackled without introducing the device (3) by the method of Dyke & Patterson (1952), using an iterative solution of maximum-likelihood equations. The present method is considerably quicker. The methods are probably asymptotically equivalent when the n_i and n_i' are all large. The transformation (12) can be applied in a similar way to more complicated problems involving Poisson variates.

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ORTHOGONAL POLYNOMIAL FITTING

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1. There are considerable advantages to be gained by using orthogonal polynomials when fitting curves to data by the method of least squares. The computer whose data are equally spaced, and of equal weight, is well catered for by tables, the best known of which are due to Fisher & Yates (1938–53), extended by Anderson & Houseman (1942); similar tables, due to Van der Reyden (1943), have been partly reproduced in the new *Biometrika Tables for Statisticians* (Pearson & Hartley, 1953). In the general case of unequally spaced and weighted data similar tables are impracticable; what the computer needs is a good calculation scheme, so designed that it makes little appeal to the memory, or the mathematics, and is as far as possible self-checking. There is only one mathematical basis for the methods, but a number of computing techniques have been devised, chiefly in order to carry out the inversion of the matrix whose elements are the equivalent of the variance and covariance estimates in the parallel problem of multiple regression. The only scheme known to the authors as having appeared in *Biometrika* is due to Isserlis (1927), but this needs revision. Elsewhere there are protagonists of the Gauss-Doolittle methods, described in more than one text-book; of the pivotal condensation method due to Aitken (1933, 1937); and of the Choleski method dealt with by Turing (1948), Fox, Huskey & Wilkinson (1948), Fox (1950), Fox & Hayes (1951), Rushton (1951), Hayes & Vickers (1951), and others. The recent papers by Guest (1950, 1953) deal with the mathematics of orthogonal polynomial fitting, and a practical computation scheme is given in the former, consisting of a combination of the Doolittle and pivotal condensation methods. (In fact, the available methods seem to differ only in the order and arrangement of the computations.)

The purpose of the present paper is to supplement the *Biometrika Tables* by offering a single recommended computation scheme for the general case. Like some other schemes, it is set out in tabular or 'blank' form, for ease of general manipulation. It is approached through multiple regression analysis, and to stress this fact the standard regression notation has been used. The arrangement is such that everything that can possibly be wanted is obtained within the tabular scheme. One point which may be considered convenient, e.g. for plotting purposes, is that while the orthogonal polynomials are calculated, the result is also given at any stage as a simple polynomial. One of the authors has examined the alternative schemes in application to the illustrative example given later in the paper, and the conclusion is that the method and scheme now offered are the most convenient for use by the average computer.

2. We begin by outlining a useful method of conducting a multiple regression analysis which brings to account one independent variate at a time, so that the successive terms of the final regression equation are orthogonal to one another, although it can in the end be converted to the usual form. Although well known, computational details of the method are not very readily accessible. They were published a few years ago by one of the authors (Wishart, 1950), while Woolf (1951) later gave practical directions for regression calculations by successive insertion of additional variates, using throughout the Gauss multipliers.

Consider a dependent variate x_n, usually assumed for significance test purposes to be normally distributed, and independent variates $x_1, x_2, \ldots, x_{n-1}$, usually assumed (for the same purposes) to be fixed; these are intended to be dealt with in the order stated. Let w denote the weight to be given to a sample observation. Add an additional 'independent' variate x_0 having a fixed value unity. Let Σ denote a sum of weighted products (or squares) of the designated variate measures taken over the sample, the size of which is N.

For $p, q = 0, 1, 2, \ldots, n$ we write

$$S_{pq} = \Sigma(wx_p x_q), \tag{2·1}$$

and then calculate in succession the partial sums

$$\left.\begin{aligned} S_{p1\cdot0} &= S_{p1} - S_{p0}S_{10}/S_{00} && (p = 1, 2, \ldots, n),\\ S_{p2\cdot01} &= S_{p2\cdot0} - S_{p1\cdot0}S_{21\cdot0}/S_{11\cdot0} && (p = 2, 3, \ldots, n),\\ S_{p3\cdot012} &= S_{p3\cdot01} - S_{p2\cdot01}S_{32\cdot01}/S_{22\cdot01} && (p = 3, 4, \ldots, n),\\ S_{p4\cdot0123} &= S_{p4\cdot012} - S_{p3\cdot012}S_{43\cdot012}/S_{33\cdot012} && (p = 4, \ldots, n), \end{aligned}\right\} \tag{2·2}$$

and so on. The displayed formulae (with $n = 4$) carry us as far as three independent variates, apart from x_0. We have in particular

$$S_{00} = \Sigma(w), \quad S_{p0} = \Sigma(wx_p).$$

The estimated regression coefficients, of progressively increasing order, are then calculated in turn, thus:

$$\left.\begin{aligned} b_{n0} &= S_{n0}/S_{00},\\ b_{n1\cdot0} &= S_{n1\cdot0}/S_{11\cdot0},\\ b_{n2\cdot01} &= S_{n2\cdot01}/S_{22\cdot01},\\ b_{n3\cdot012} &= S_{n3\cdot012}/S_{33\cdot012}, \end{aligned}\right\} \tag{2·3}$$

and so on. In particular $b_{n0} = \Sigma(wx_n)/\Sigma(w) = \bar{x}_n$.

Partial x-variates form the next sequence, and are calculated as below:

$$\left.\begin{aligned} x_{p\cdot0} &= x_p - S_{p0}x_0/S_{00},\\ x_{p\cdot01} &= x_{p\cdot0} - S_{p1\cdot0}x_{1\cdot0}/S_{11\cdot0},\\ x_{p\cdot012} &= x_{p\cdot01} - S_{p2\cdot01}x_{2\cdot01}/S_{22\cdot01}, \end{aligned}\right\} \tag{2·4}$$

and so on. The first of these is $x_{p\cdot0} = x_p - \bar{x}_p$.

The multiple regression equation, written as a sum of orthogonal components, is then

$$X_n = b_{n0}x_0 + b_{n1\cdot0}x_{1\cdot0} + b_{n2\cdot01}x_{2\cdot01} + b_{n3\cdot012}x_{3\cdot012} + \ldots, \tag{2·5}$$

of which the first term on the right-hand side is $\bar{x}_n$. For each term fitted there is a 'sum of squares' due to the regression, with one degree of freedom and, when all have been fitted to the chosen number of independent variates, there is a residual sum of squares. The analysis of variance, taken up to three variates, is shown in Table 1; in this, on the usual assumptions, the various mean squares are distributed as variance estimates independently of one another, with the assigned numbers of degrees of freedom. In the table the contribution due to $\bar{x}_n$ has been kept separate from S_{nn}; in the more usual form the total sum of squares would be $S_{nn\cdot0}$, with $N-1$ degrees of freedom. The overall variance-ratio test of regression after bringing in three variates is illustrated, but note that this test could have been applied at an earlier stage, also that at any stage the mean square due to a single regression coefficient is tested against the residual mean square available at that stage.

Table 1

Regression coefficient	Degrees of freedom	Sums of squares		Mean squares
b_{n0}	1	$b_{n0}S_{n0}$	$=S_{nn}-S_{nn\cdot0}$	
$b_{n1\cdot0}$	1	$b_{n1\cdot0}S_{n1\cdot0}$	$=S_{nn\cdot0}-S_{nn\cdot01}$	
$b_{n2\cdot01}$	1	$b_{n2\cdot01}S_{n2\cdot01}$	$=S_{nn\cdot01}-S_{nn\cdot012}$	$S_{nn\cdot0}-S_{nn\cdot0123}$ V_b
$b_{n3\cdot012}$	1	$b_{n3\cdot012}S_{n3\cdot012}$	$=S_{nn\cdot012}-S_{nn\cdot0123}$	
Residual	$N-4$	$S_{nn\cdot0123}$		V_r
Total	N	S_{nn}		

In practice it is helpful to choose as x_1 that variate which is most nearly expected to be associated with x_n; then as x_2 the next in expected strength of association, and so on. The calculations can be reviewed when a fitted term is found to be insignificant; the appropriate variate may be discarded and another substituted. As a result of this procedure it may be possible to say with a minimum amount of labour that x_n is adequately 'predicted' by a limited number only of independent variates, a matter of some importance in a variety of problems, for example, in physical anthropology where large numbers of body and other measurements may be available, and for practical reasons it is desirable to deal with a small number of variates only for purposes of description.

The more symmetrical arrangement of the regression equation, giving all the partial regression coefficients of the highest order, and also tests of their individual significance and a final check, require additional calculations which can be readily inferred from the application of the method which is made in §3 to orthogonal polynomial fitting.

It will be noted that the first stage in the scheme involves the determination of the weighted sums of products (and squares) of deviations of variate measures from their (weighted) means. That is, $S_{pq\cdot0}$ is determined from S_{pq} by removing the usual 'correction factor' $S_{p0}S_{q0}/S_{00}$. If the quantities $S_{pq\cdot0}$ are given in the first instance, one stage in the computations is eliminated.

3. Turning now to the problem of orthogonal polynomial fitting, we read x_n as y, to distinguish the dependent variate. We then let

$$x_0 = x^0 = 1, \quad x_1 = x^1 = x, \quad x_2 = x^2, \quad \ldots,$$

so that the independent variates become the powers of x, including the dummy variate. In the special case of $n = N$, the sample size, so that there are $N-1$ independent variates, the multiple regression equation will then become the polynomial of degree $N-1$ of perfect fit. The usual case is, however, where we endeavour to fit a polynomial of degree p, where $p < N-1$. We then have

$$S_{r+s,0} = S_{r+s-1,1} = \ldots = S_{rs} = \ldots = S_{0,r+s} = \Sigma(wx^{r+s}), \tag{3·1}$$

where r and s take independently the values $0, 1, 2, \ldots, p$. We define also

$$S_{yr} = \Sigma(wyx^r), \text{ needed for } r = 0, 1, 2, \ldots, p. \tag{3·2}$$

Equations (2·3) will be needed for $n = 1, 2, \ldots, p$ as well as for $n = y$, and we shall also find it convenient to calculate the quantities $b_{20\cdot1}, b_{30\cdot12}, \ldots$ which are defined below. In particular b_{r0} is the mean value of the rth powers of the x's.

In virtue of (2·3), equations (2·2) may be calculated as

$$\left.\begin{aligned} S_{r1\cdot0} &= S_{r1} - b_{r0}S_{10} && (r = 1, 2, \ldots, p \text{ and } y),\\ S_{r2\cdot01} &= S_{r2\cdot0} - b_{r1\cdot0}S_{21\cdot0} && (r = 2, 3, \ldots, p \text{ and } y),\\ S_{r3\cdot012} &= S_{r3\cdot01} - b_{r2\cdot01}S_{32\cdot01} && (r = 3, \ldots, p \text{ and } y), \end{aligned}\right\} \tag{3·3}$$

and so on. Equation (2·5) becomes, in the present case,

$$Y = b_{y0}x_0 + b_{y1\cdot0}x_{1\cdot0} + b_{y2\cdot01}x_{2\cdot01} + b_{y3\cdot012}x_{3\cdot012} + \ldots. \tag{3·4}$$

In Fisher's notation (Fisher, 1925) the successive orthogonal polynomials are $\xi_0 = 1, \xi_1, \xi_2, \ldots$, with multipliers $A, B, C, \ldots$, so that the fitted polynomial is

$$Y = A + B\xi_1 + C\xi_2 + D\xi_3 + \ldots. \tag{3·5}$$

From equations (2·4), by inserting b-coefficients, we have

$$\left.\begin{aligned} x_{1\cdot0} &= x_1 - b_{10}, \\ x_{2\cdot01} &= x_{2\cdot0} - b_{21\cdot0}x_{1\cdot0}, \\ x_{3\cdot012} &= x_{3\cdot01} - b_{32\cdot01}x_{2\cdot01}, \end{aligned}\right\} \tag{3·6}$$

and so on. In virtue of the relations

$$\left.\begin{aligned} \bar{x} = b_{10}, \quad \overline{x^2} = b_{20}, \quad \overline{x^3} = b_{30}, \ldots; \\ b_{20} - b_{21\cdot0}b_{10} = b_{20\cdot1}; \\ b_{30} - b_{31\cdot0}b_{10} - b_{32\cdot01}b_{20\cdot1} = b_{30\cdot12}; \\ b_{31\cdot0} - b_{32\cdot01}b_{21\cdot0} = b_{31\cdot02}, \end{aligned}\right\} \tag{3·7}$$

and so on, we find, after appropriate substitution and simplification, replacing $x_0, x_1, x_2, \ldots$ by $x^0, x^1, x^2, \ldots$, that

$$\left.\begin{aligned} x_0 &= \xi_0 = 1, \\ x_{1\cdot0} &= \xi_1 = -b_{10} + x, \\ x_{2\cdot01} &= \xi_2 = -b_{20\cdot1} - b_{21\cdot0}x + x^2, \\ x_{3\cdot012} &= \xi_3 = -b_{30\cdot12} - b_{31\cdot02}x - b_{32\cdot01}x^2 + x^3, \end{aligned}\right\} \tag{3·8}$$

and so on, and that

$$A = b_{y0}, \quad B = b_{y1\cdot0}, \quad C = b_{y2\cdot01}, \quad D = b_{y3\cdot012}, \ldots. \tag{3·9}$$

By combining we may write down the different terms of Y separately, or they may be cumulated one at a time, thus having the appropriate fitted curve at any stage expressed as a simple polynomial in x, as follows:

$$\left.\begin{aligned} Y_0 &= b_{y0}, \\ Y_1 &= b_{y0\cdot1} + b_{y1\cdot0}x, \\ Y_2 &= b_{y0\cdot12} + b_{y1\cdot02}x + b_{y2\cdot01}x^2, \\ Y_3 &= b_{y0\cdot123} + b_{y1\cdot023}x + b_{y2\cdot013}x^2 + b_{y3\cdot012}x^3, \end{aligned}\right\} \tag{3·10}$$

and so on, where Y_r denotes (3·4) taken to degree r. The coefficients are calculated from (2·3), putting $n = y$, and from (3·7) with y written for $2, 3, \ldots$.

The analysis of variance will be as in Table 1 with y substituted for n. The estimated variances of the coefficients of the power series in (3·10) may be written down in terms of the following 'Gauss multipliers':

$$\left.\begin{aligned} c_{00} &= S_{00}^{-1}, \\ c_{00\cdot1} &= c_{00} - b_{10}c_{10}, & c_{11\cdot0} &= S_{11\cdot0}^{-1}, \\ c_{00\cdot12} &= c_{00\cdot1} - b_{20\cdot1}c_{20\cdot1}, & c_{11\cdot02} &= c_{11\cdot0} - b_{21\cdot0}c_{21\cdot0}, \\ c_{00\cdot123} &= c_{00\cdot12} - b_{30\cdot12}c_{30\cdot12}, & c_{11\cdot023} &= c_{11\cdot02} - b_{31\cdot02}c_{31\cdot02}, \\ c_{22\cdot01} &= S_{22\cdot01}^{-1}, \\ c_{22\cdot013} &= c_{22\cdot01} - b_{32\cdot01}c_{32\cdot01}, & c_{33\cdot012} &= S_{33\cdot012}^{-1}, \end{aligned}\right\} \tag{3·11}$$

in which

$$c_{10} = -b_{10}c_{11\cdot0}, \quad c_{20\cdot1} = -b_{20\cdot1}c_{22\cdot01}, \quad c_{21\cdot0} = -b_{21\cdot0}c_{22\cdot01}, \quad c_{30\cdot12} = -b_{30\cdot12}c_{33\cdot012},$$
$$c_{31\cdot02} = -b_{31\cdot02}c_{33\cdot012}, \quad c_{32\cdot01} = -b_{32\cdot01}c_{33\cdot012},$$

and so on. If we denote the 'mean squares' by

$$S_{yy\cdot0}/(N-1) = s_0^2, \quad S_{yy\cdot01}/(N-2) = s_1^2, \ldots,$$

then, V denoting 'estimated variance of', we have

$$\left.\begin{aligned} V(b_{y0}) &= c_{00}s_0^2; \\ V(b_{y0\cdot1}) &= c_{00\cdot1}s_1^2, \quad V(b_{y1\cdot0}) = c_{11\cdot0}s_1^2, \\ V(b_{y0\cdot12}) &= c_{00\cdot12}s_2^2, \quad V(b_{y1\cdot02}) = c_{11\cdot02}s_2^2, \\ V(b_{y0\cdot123}) &= c_{00\cdot123}s_3^2, \quad V(b_{y1\cdot023}) = c_{11\cdot023}s_3^2, \\ V(b_{y2\cdot01}) &= c_{22\cdot01}s_2^2; \\ V(b_{y2\cdot013}) &= c_{22\cdot013}s_3^2, \quad V(b_{y3\cdot012}) = c_{33\cdot012}s_3^2, \end{aligned}\right\} \tag{3·12}$$

and so on. Finally, it may be useful to have formulae for the estimated variances of the ordinates of the different order curves. Calling these ordinates $\hat{Y}_0, \hat{Y}_1, \ldots$, we have

$$\left.\begin{aligned} V(\hat{Y}_0) &= c_{00}\xi_0^2 s_0^2, \\ V(\hat{Y}_1) &= (c_{00}\xi_0^2 + c_{11\cdot0}\xi_1^2)s_1^2, \\ V(\hat{Y}_2) &= (c_{00}\xi_0^2 + c_{11\cdot0}\xi_1^2 + c_{22\cdot01}\xi_2^2)s_2^2, \\ V(\hat{Y}_3) &= (c_{00}\xi_0^2 + c_{11\cdot0}\xi_1^2 + c_{22\cdot01}\xi_2^2 + c_{33\cdot012}\xi_3^2)s_3^2, \end{aligned}\right\} \tag{3·13}$$

and so on, in which, of course, $\xi_0 = 1$ and the values of $\xi_1, \xi_2, \ldots$ are obtained from (3·8) for any value of x.

4. For practical computation purposes it is desirable to systematize the evaluation of the quantities set out in succession in §3. This leads to the tabular scheme set out in Table 2, which proceeds as far as the fitting of a cubic, and can readily be extended for higher degree polynomials. Owing to the symmetry of the original matrix of sums of squares and products, one need only enter quantities on and to the right of the diagonals. This is indicated by the heavy vertical lines in the upper part of the table. A small number of supplementary lines are needed to obtain the actual polynomial coefficients; these are on the left immediately below the main table. The column on the right is used for the various contributions to the sum of squares, and from this the analysis of variance is readily written down. The check column is of the usual character for this type of work. The operations of the rest of the table are followed here, and then in those lines where a tick mark is shown the check is that the number in this column is the sum of all numbers in that line to the left, stopping at the heavy vertical rule. A useful final check is shown in the lower box on the right. The table shown in lines 1′–16′ is a subsidiary one, giving the matrix of the c's, inverse to that of the S's. Only the diagonal c's are required for the estimated variances of (3·12) and (3·13), but the whole matrix has been included for the sake of completeness. It is useful to have all the c's readily available, particularly when the scheme is used for multiple regression (see §2), since they are needed in calculating adjusted coefficients of the surviving independent variates, without redoing the calculations, when it has been decided to omit any such variate. The appropriate formulae for this purpose were given by R. A. Fisher in the 5th edition of *Statistical Methods for Research Workers*, and discussed by Cochran (1938).

Table 2

						Check column	Sums of squares
1	S_{00}	S_{10}	S_{20}	S_{30}	S_{y0}		S_{yy}
2	-1	$-b_{10}$	$-b_{20}$	$-b_{30}$	$-b_{y0}$	✓	$b_{y0}S_{y0}$
3		S_{11}	S_{21}	S_{31}	S_{y1}		
4	$-b_{10}$	$-b_{10}S_{10}$	$-b_{10}S_{20}$	$-b_{10}S_{30}$	$-b_{10}S_{y0}$		
5	$-b_{10}$	$S_{11\cdot0}$	$S_{21\cdot0}$	$S_{31\cdot0}$	$S_{y1\cdot0}$	✓	
6	c_{10}	-1	$-b_{21\cdot0}$	$-b_{31\cdot0}$	$-b_{y1\cdot0}$	✓	$b_{y1\cdot0}S_{y1\cdot0}$
7			S_{22}	S_{32}	S_{y2}		
8	$-b_{20}$		$-b_{20}S_{20}$	$-b_{20}S_{30}$	$-b_{20}S_{y0}$		
9	$b_{21\cdot0}b_{10}$	$-b_{21\cdot0}$	$-b_{21\cdot0}S_{21\cdot0}$	$-b_{21\cdot0}S_{31\cdot0}$	$-b_{21\cdot0}S_{y1\cdot0}$		
10	$-b_{20\cdot1}$	$-b_{21\cdot0}$	$S_{22\cdot01}$	$S_{32\cdot01}$	$S_{y2\cdot01}$	✓	
11	$c_{20\cdot1}$	$c_{21\cdot0}$	-1	$-b_{32\cdot01}$	$-b_{y2\cdot01}$	✓	$b_{y2\cdot01}S_{y2\cdot01}$
12				S_{33}	S_{y3}		
13	$-b_{30}$			$-b_{30}S_{30}$	$-b_{30}S_{y0}$		
14	$b_{31\cdot0}b_{10}$	$-b_{31\cdot0}$		$-b_{31\cdot0}S_{31\cdot0}$	$-b_{31\cdot0}S_{y1\cdot0}$		
15	$b_{32\cdot01}b_{20\cdot1}$	$b_{32\cdot01}b_{21\cdot0}$	$-b_{32\cdot01}$	$-b_{32\cdot01}S_{32\cdot01}$	$-b_{32\cdot01}S_{y2\cdot01}$		
16	$-b_{30\cdot12}$	$-b_{31\cdot02}$	$-b_{32\cdot01}$	$S_{33\cdot012}$	$S_{y3\cdot012}$	✓	
17	$c_{30\cdot12}$	$c_{31\cdot02}$	$c_{32\cdot01}$	-1	$-b_{y3\cdot012}$	✓	$b_{y3\cdot012}S_{y3\cdot012}$
18	b_{y0}						$S_{yy}-S_{yy\cdot0123}$
19	$-b_{10}b_{y1\cdot0}$	$b_{y1\cdot0}$					
20	$b_{y0\cdot1}$	$b_{y1\cdot0}$					
21	$-b_{20\cdot1}b_{y2\cdot01}$	$-b_{21\cdot0}b_{y2\cdot01}$	$b_{y2\cdot01}$				$b_{y0\cdot123}S_{y0}$
22	$b_{y0\cdot12}$	$b_{y1\cdot02}$	$b_{y2\cdot01}$				$+b_{y1\cdot023}S_{y1}$
23	$-b_{30\cdot12}b_{y3\cdot012}$	$-b_{31\cdot02}b_{y3\cdot012}$	$-b_{32\cdot01}b_{y3\cdot012}$	$b_{y3\cdot012}$			$+b_{y2\cdot013}S_{y2}$
24	$b_{y0\cdot123}$	$b_{y1\cdot023}$	$b_{y2\cdot013}$	$b_{y3\cdot012}$			$+b_{y3\cdot012}S_{y3}$
							$=S_{yy}-S_{yy\cdot0123}$

Final check

1′	c_{00}			
2′	$-b_{10}c_{10}$			
3′	$c_{00\cdot1}$			
4′	$-b_{20\cdot1}c_{20\cdot1}$			
5′	$c_{00\cdot12}$			
6′	$-b_{30\cdot12}c_{30\cdot12}$			
7′	$c_{00\cdot123}$			
8′	c_{10}	$c_{11\cdot0}$		
9′	$-b_{21\cdot0}c_{20\cdot1}$	$-b_{21\cdot0}c_{21\cdot0}$		
10′	$c_{10\cdot2}$	$c_{11\cdot02}$		
11′	$-b_{31\cdot02}c_{30\cdot12}$	$-b_{31\cdot02}c_{31\cdot02}$		
12′	$c_{10\cdot23}$	$c_{11\cdot023}$		
13′	$c_{20\cdot1}$	$c_{21\cdot0}$	$c_{22\cdot01}$	
14′	$-b_{32\cdot01}c_{30\cdot12}$	$-b_{32\cdot01}c_{31\cdot02}$	$-b_{32\cdot01}c_{32\cdot01}$	
15′	$c_{20\cdot13}$	$c_{21\cdot03}$	$c_{22\cdot013}$	
16′	$c_{20\cdot13}$	$c_{21\cdot03}$	$c_{22\cdot01}$	$c_{23\cdot012}$

Subsidiary table giving the c's.

Table 3

1	14	91	803	7705	78	8691	538
2	−1	−6·5	−57·357143	−550·357143	−5·571429	−620·78571 ✓	434·5715
3		803	7705	77447	589	86635	
4	−6·5	−591·5	−5219·5	−50082·5	−507	−56491·5	
5	−6·5	211·5	2485·5	27364·5	82	30143·5 ✓	
6	−0·0307329	−1	−11·751773	−129·382979	−0·387707	−142·52246 ✓	31·7920
7			77447	804121	5049	895125	
8	−57·357143		−46057·7858	−441936·787	−4473·8573	−498490·930	
9	76·386525	−11·751773	−29209·0318	−321581·392	−963·6454	−354239·569	
10	19·029382	−11·751773	2180·1824	40602·821	−388·5027	42394·501 ✓	
11	0·00872834	−0·00539027	−1	−18·623589	0·178197	−19·44539 ✓	69·2300
12				8555663	46207	9491143	
13	−550·357143			−4240501·8	−42927·857	−4783153·93	
14	840·989364	−129·382979		−3540500·5	−10609·404	−3900055·83	
15	−354·395389	218·860190	−18·623589	−756170·3	7235·315	−789537·76	
16	−63·763168	89·477211	−18·623589	18490·4	−94·946	18395·48 ✓	
17	−0·00344845	0·00483911	−0·00100720	−1	0·005135	−0·994865 ✓	0·4875
18	5·571429						536·0810
19	−2·520096	0·387707					
20	3·051333	0·387707					−0·9533
21	−3·390979	2·094131	−0·178197				1191·1777
22	−0·339646	2·481838	−0·178197				−416·8707
23	0·327424	−0·459465	0·095632	−0·005135			−237·2729
24	−0·012222	2·022373	−0·082565	−0·005135			536·0808

1′	0·0714286			
2′	0·1997638			
3′	0·2711924			
4′	0·1660949			
5′	0·4372873			
6′	0·2198841			
7′	0·6571714			
8′	−0·0307329	0·0047281		
9′	−0·1025736	0·0633452		
10′	−0·1333065	0·0680733		
11′	−0·3085577	0·4329902		
12′	−0·4418642	0·5010635		
13′	0·00872834	−0·00539027	0·000458677	
14′	0·06422252	−0·09012160	0·018757678	
15′	0·07295086	−0·09551187	0·019216355	
16′	−0·00344845	0·00483911	−0·00100720	0·0000540821

Little explanation is needed of the operation of this scheme. The primary S's (from (3·1) and (3·2)) are entered in lines 1, 3, 7 and 12. To form line 2, all quantities in line 1 are divided by S_{00}, whose reciprocal is c_{00}, a quantity which is at the same time entered on line 1′. In this and subsequent calculations the signs are changed to the right of the heavy vertical rule. Line 6 is similarly obtained from line 5, line 11 from line 10 and line 17 from line 16, by dividing by $S_{11\cdot0}$, $S_{22\cdot01}$ and $S_{33\cdot012}$, whose reciprocals are $c_{11\cdot0}$, $c_{22\cdot01}$ and $c_{33\cdot012}$ respectively. Line 4 comes from line 1 as shown using the multiplier $-b_{10}$; the next line is the sum of lines 3 and 4, as is indicated by the short horizontal rules. At the next stage lines 8 and 9 are derived from lines 1 and 5 by the use of b-multipliers already determined in lines 2 and 6; line 10 is another summation, this time of lines 7, 8 and 9. And so on. The appropriate b's and c's are entered in the supplementary lines and subsidiary table as calculated, the subsequent calculations in these lines being as shown; vertical additions are made in pairs of lines, for example line 20 is the sum of lines 18 and 19, line 3′ is the sum of lines 1′ and 2′, and so on.

We are only concerned in this paper with the fitting of orthogonal polynomials, and the computation scheme has therefore been drawn up with this end in view. But it should be pointed out that the scheme generally is one for matrix inversion, at any rate where the simultaneous equations whose solution is desired are symmetrical, and it may be found to compare favourably with other schemes which have been advocated in the past. It requires no adaptation to be used in standard multiple regression work, as described in §2.

5. *Numerical example.* We shall fit as far as the third degree polynomial the following data:

Weight (w)	1	2	1	1	2	3	1	2	1
Values of x	0	1	3	4	7	8	10	11	12
Values of y	0	2	5	6	9	8	7	5	4

Since for a polynomial of the pth degree we require the weighted sums of the first $2p$ powers of x and the weighted sums of products of y with the first p powers of x, as well as the weighted sum of squares of y (for the analysis of variance), the first step consists in calculating these quantities, either by means of a tabular build-up or by using tables of powers. The calculation yields

$$
\begin{aligned}
S_{00} &= \Sigma(w) = 14 & S_{y0} &= \Sigma(wy) = 78\\
S_{10} = S_{01} &= \Sigma(wx) = 91 & S_{y1} &= \Sigma(wyx) = 589\\
S_{20} = S_{11} = S_{02} &= \Sigma(wx^2) = 803 & S_{y2} &= \Sigma(wyx^2) = 5049\\
S_{30} = S_{21} = S_{12} = S_{03} &= \Sigma(wx^3) = 7705 & S_{y3} &= \Sigma(wyx^3) = 46207\\
S_{31} = S_{22} = S_{13} &= \Sigma(wx^4) = 77447 & &\\
S_{32} = S_{23} &= \Sigma(wx^5) = 804121 & &\\
S_{33} &= \Sigma(wx^6) = 8555663 & S_{yy} &= \Sigma(wy^2) = 538
\end{aligned}
$$

These quantities are now entered in the appropriate places of Table 2, and calculation proceeds. This is shown in Table 3, in which enough decimal places have been retained to show agreement at the final check to two or more places of decimals. Fewer figures need be written down if the accuracy desired is not of this order. For all ordinary purposes a maximum of four decimal places, or perhaps four significant figures, will suffice.

The fitted curves of degrees 0, 1, 2 and 3 are:

$$Y_0 = 5{\cdot}57143,$$
$$Y_1 = 3{\cdot}05133 + 0{\cdot}38771x,$$
$$Y_2 = -0{\cdot}33965 + 2{\cdot}48184x - 0{\cdot}17820x^2,$$
$$Y_3 = -0{\cdot}01222 + 2{\cdot}02237x - 0{\cdot}08256x^2 - 0{\cdot}00514x^3.$$

The analysis of variance need not be separately written out. The residual sum of squares, from Table 3, is $538 - 536{\cdot}0810 = 1{\cdot}9190$, with 5 degrees of freedom. s_3^2 is therefore 0·3838, whence it follows that the linear and quadratic terms, with mean squares 31·7920 and 69·2300, are significant at the 0·1 % point. The cubic term, with mean square 0·4875, is not significant. If this were a practical problem we should either go on to one more stage, or, if that was not thought worth while, rest content with describing the data by means of the above quadratic Y_2. This may be plotted against the actual data by substituting for x, when we get the following results:

Weight (w)	1	2	1	1	2	3	1	2	1
Values of y	0	2	5	6	9	8	7	5	4
Values of Y_2	−0·34	1·96	5·50	6·74	8·30	8·11	6·66	5·40	3·78

s_2^2 will be $(0{\cdot}4875 + 1{\cdot}9190)/6 = 2{\cdot}4065/6 = 0{\cdot}4011$. This is multiplied by $c_{00\cdot12}$, $c_{11\cdot02}$ and $c_{22\cdot01}$ respectively, and the square roots taken, to deduce the standard errors of the coefficients $b_{y0\cdot12}$, $b_{y1\cdot02}$ and $b_{y2\cdot01}$ respectively, of Y_2. These standard errors are 0·4188, 0·1652 and 0·0136 respectively. From these it appears that $b_{y1\cdot02}$ and $b_{y2\cdot01}$ are large in comparison with their standard errors, and very significant, but that $b_{y0\cdot12}$ is not significantly different from zero. The calculation of the estimated standard error of an estimated Y_2 may be illustrated by taking the value 6·74, which corresponds to $x = 4$. $s_2^2 = 0{\cdot}4011$ has to be multiplied, as in (3·13), by

$$c_{00}\xi_0^2 + c_{11\cdot0}\xi_1^2 + c_{22\cdot01}\xi_2^2,$$

in which the c's are obtained from Table 3 and the ξ's are given in (3·8) in terms of the b's, also obtained from Table 3, and x. Putting $x = 4$, multiplying the above expression by 0·4011 and extracting the square root, we find the estimated standard error at this point to be 0·26.

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POPULATION DIFFERENCES BETWEEN SPECIES GROWING ACCORDING TO SIMPLE BIRTH AND DEATH PROCESSES

By J. H. DARWIN

1. Introduction

In some ecological problems, treated especially by C. B. Williams (e.g. 1944), the numerical sizes of different species of plants or animals present in a sampling region have followed Fisher's logarithmic law, viz. the expected number of species having an observed population size n is $\alpha x^n/n$ $(n = 1, 2, \ldots)$. In this α is a constant depending only on the biological association being sampled and not on the interval of sampling. This law for the observed species was derived from the negative binomial distribution, for which the probability that a species has n members is given by

$$P_n = \frac{\Gamma(n+k)}{\Gamma(n+1)\,\Gamma(k)} \frac{\sigma^n}{(1+\sigma)^{n+k}} \quad (n = 0, 1, \ldots),$$

by allowing the parameter k to tend to 0, and the number N of possible species to tend to infinity in such a way that $Nk = \alpha$, $\sigma/(1+\sigma) = x$. The negative binomial is a compound Poisson distribution, i.e. it arises from the assumption that the population of each species is represented by a Poisson distribution with mean m, the different m's for the different species being regarded as a random sample from the distribution

$$(1/\sigma^k \Gamma(k))\, x^{k-1} e^{-x/\sigma}\, dx.$$

Thus if there is a large number of species, mostly rare, in danger of losing a member to the sample, Fisher's logarithmic law might be a reasonable description of the heterogeneity between them.

One may further discuss this heterogeneity in terms of the manner in which different species grow in the sampling region. For instance, D. G. Kendall (1948*a*) considered a population expanding in a simple birth and death process in which the probabilities of a birth or a death from among n members alive at time t are, for the next interval dt, $n\lambda\,dt$ and $n\mu\,dt$ respectively; in dt there is also a probability $\kappa\,dt$ of the entry into the region of a new member from outside; λ, μ and κ are constants. These transition probabilities lead to the negative binomial as a current description of the size of a species whose first member in the region is an immigrant. When the immigration constant κ is small compared with λ and μ, a logarithmic law with a zero class results. The often observed logarithmic type differences in specific population size would then be reasonable if the species come into the region at times randomly scattered in the interval $(0, t)$, and grow roughly according to the same geometric birth and death process. The possibilities of such 'explanations' of heterogeneity between species can be formally investigated for different theoretical distributions. It will be shown that when the population size is bounded, the form of solution of processes for which the transition probability rates do not vary with time enables one to test if a particular distribution can describe the state of the process at any time. When the transition probability rates do vary with time, it appears that there may be many stochastic geneses of such well-known distributions as the negative binomial and the Poisson. Finally, a simple evolutionary

model for the changing number of species is applied to species which expand geometrically, and logarithmic type distributions of the observed population sizes of the species again occur.

2. FINITE DISTRIBUTIONS

If the size of the population can take any of the values $0, 1, \ldots, N$, the equations of the birth and death process are

$$\left.\begin{aligned} \frac{dP_n}{dt} &= \alpha_{n+1}P_{n+1} - (\alpha_n + \beta_n)P_n + \beta_{n-1}P_{n-1} \quad (n = 1, 2, \ldots, N-1), \\ \frac{dP_0}{dt} &= \alpha_1 P_1 - \beta_0 P_0, \quad \frac{dP_N}{dt} = -\alpha_N P_N + \beta_{N-1}P_{N-1}, \end{aligned}\right\} \tag{1}$$

in which P_n is the probability that a species has size n (n could be the number of regions out of a possible N in which the species could be found); α_n is the probability death-rate and β_n the probability birth-rate when the size is n.

2·1. *Transition probability rates not varying with time.* When α_n and β_n are constants, which is, for instance, reasonable when the spores or seeds of a plant or the offspring of an animal can travel over distances of the same order as the dimensions of a region, the set of equations can be written as

$$\frac{d\mathbf{P}}{dt} = -A\mathbf{P}, \tag{2}$$

where $\mathbf{P}$ is the column vector of probabilities $P_0, P_1, \ldots, P_N$, and A is the matrix with non-zero elements only in the three main diagonals

$$\begin{pmatrix} \beta_0 & -\alpha_1 & & & \\ -\beta_0 & \alpha_1+\beta_1 & -\alpha_2 & & \\ & -\beta_1 & \ddots & & \\ & & & \alpha_{N-1}+\beta_{N-1} & -\alpha_N \\ & & & -\beta_{N-1} & \alpha_N \end{pmatrix}.$$

It is well known that if the latent roots λ_i ($i = 0, 1, \ldots, N$) of this matrix are distinct, the solution P_n is of the form $\sum_{i=0}^{N} A_{n_i} e^{-\lambda_i t}$, where the A_{n_i} are constants independent of time and embracing the initial conditions. It is further possible to show that if $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_{N-1}$ are > 0, these roots λ_i are real, distinct and positive (a result also established by B. J. Prendiville, unpublished). This follows from an expansion of the leading $(n+1) \times (n+1)$ minor of $|\lambda I - A|$ as $D_n(\lambda)$ in the form

$$D_n(\lambda) = (\lambda - (\alpha_n + \beta_n))\, D_{n-1}(\lambda) - \alpha_n \beta_{n-1} D_{n-2}(\lambda). \tag{3}$$

An induction hypothesis that $D_{n-1}(\lambda)$ and $D_{n-2}(\lambda)$ have real distinct roots $\mu_0, \mu_1, \ldots, \mu_{n-1}$; $\nu_0, \nu_1, \ldots, \nu_{n-2}$, such that $\mu_0 < \nu_0 < \mu_1 < \ldots < \nu_{n-2} < \mu_{n-1}$ leads to the conclusion that $D_n(\lambda)$ has real distinct roots separating those of $D_{n-1}(\lambda)$. An expansion of $|\lambda I - A|$ shows, by the sign rule, that there can be no real negative roots, and that the lowest root of $D_n(\lambda)$ for $n < N$ is > 0. The result then follows. It can be used as a criterion for testing if possible processes of this simple type could lead to given distributions.

Thus, if

$$\frac{dP_n}{dt} = P_n g_n(t) \quad \text{and} \quad P_{n+1} = r(t) P_n,$$

(1) becomes, when $P(t)$ is not identically zero,

$$g_n(t) = \alpha_{n+1} r_n - (\alpha_n + \beta_n) + \beta_{n-1}/r_{n-1}. \tag{4}$$

This is a set of $N+1$ equations in the $N+1$ coefficients of $e^{-\lambda_i t}$.

2·2. *Examples.* In the binomial, $P_n = \binom{N}{n} p^n (1-p)^{N-n}$, since $P_N = p^N$ must be of the form $\sum_{i=0}^{N} A_{n_i} e^{-\lambda_i t}$, p can only have the form $a + b\,e^{-\mu t}$, with $0 \leqslant a+b \leqslant 1$ and $0 \leqslant a \leqslant 1$. Then

$$\begin{aligned}\frac{dP_n}{dt} &= \binom{N}{n} [np^{n-1}(1-p)^{N-n} - (N-n)p^n(1-p)^{N-n-1}](-b\mu\, e^{-\mu t}) \\ &= -\mu b\, e^{-\mu t}\left(\frac{n}{p} - \frac{N-n}{1-p}\right) P_n\end{aligned}$$

and

$$r_n = \frac{N-n}{n+1}\frac{p}{1-p}.$$

Whether or not a solution is possible can be seen by substitution in (4). This produces the sets of equations

$$\alpha_{n+1}\frac{N-n}{n+1}a^2 - (\alpha_n+\beta_n)a(1-a) + \beta_{n-1}\frac{n}{N-(n-1)}(1-a)^2 = 0, \tag{5}$$

$$\alpha_{n+1}\frac{N-n}{n+1}2a - (\alpha_n+\beta_n)(1-2a) - \beta_{n-1}\frac{n}{N-(n-1)}2(1-a) = \mu(Na-n), \tag{6}$$

$$\alpha_{n+1}\frac{N-n}{n+1} + (\alpha_n+\beta_n) + \beta_{n-1}\frac{n}{N-(n-1)} = \mu N. \tag{7}$$

These provide $3(N+1)$ equations for $2(N+1)$ unknowns α_n, β_n, but the sets are not independent. The unique solution satisfying them is the well-known one

$$\alpha_n = \alpha n, \quad \beta_n = \beta(N-n); \qquad a = \beta/(\alpha+\beta), \quad \mu = (\alpha+\beta),$$

and b is determined by the initial conditions.

Polya (1930) considered the compound binomial distribution

$$\begin{aligned}P_n &= \int_0^1 \binom{N}{n} p^n (1-p)^{N-n} \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}\,dp \\ &= \binom{N}{n} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\,\beta(\beta+1)\ldots(\beta+N-n-1)}{(\alpha+\beta)(\alpha+\beta+1)\ldots(\alpha+\beta+N-1)}.\end{aligned} \tag{8}$$

In his work it arose from a contagion urn scheme in which, after each of N drawings from an urn containing α white balls and β black, two balls of the colour drawn are replaced. P_n is then the probability of obtaining n white balls. Skellam (1948) applies the distribution to the secondary association of chromosomes in *Brassica*, and to accident and traffic problems. It can also be used as a description of the spread of species over N regions when the population size of a species increases by one for each new region in which it is found. The heterogeneity in the binomial then describes that between the abundances of the different species.

The form of P_n, as far as the constants α and β are concerned, is as in a hypergeometric distribution, so that the present result will also apply to that.

The only rational form of α and β that will not make P_n an infinite series in powers of some $e^{-\mu t}$ is $\alpha = a - c\,e^{-kt}$ and $\beta = b + c\,e^{-kt}$, where $0 \leqslant a, b \leqslant 1$, and $c \leqslant a$, $1-b$ if $c \geqslant 0$, and $-c \leqslant b$, $1-a$ if $c \leqslant 0$. Such an assumption is consistent, for instance, with the spread of a species from a few regions to many, or with the reverse process.

The equations similar to (5), (6) and (7) are now $N+1$ in number, the simplest to find coming from the coefficients of e^{-0kt}, e^{-Nkt} and e^{-kt}. They are:

$$\alpha_{n+1}(a+n)(a+n-1)(N-n)(N-(n-1))$$
$$-(\alpha_n+\beta_n)(n+1)(N-(n-1))(b+N-(n+1))(a+n-1)$$
$$+\beta_{n-1}n(n+1)(b+N-n)(b+N-(n-1))=0, \quad (9)$$

$$\alpha_{n+1}(N-n)(N-(n-1))+(\alpha_n+\beta_n)(n+1)(N-(n-1))+\beta_{n-1}n(n+1)$$
$$=-k(n+1)(N-(n-1)), \quad (10)$$

$$\alpha_{n+1}(N-n)(N-(n-1))(a+n)(a+n-1)[\psi(a+n-1)-\psi(a)-\psi(b+N-(n+1))+\psi(b)]$$
$$-(\alpha_n+\beta_n)(n+1)(N-(n-1))(a+n-1)(b+N-(n+1))$$
$$\times[\psi(a+n)-\psi(a)-\psi(b+N-n)+\psi(b)]+\beta_{n-1}n(n+1)(b+N-n)(b+N-(n-1))$$
$$\times[\psi(a+n-1)-\psi(a)-\psi(b+N-(n-1))+\psi(b)]$$
$$=-k(n+1)(N-(n-1))(b+N-(n+1))(a+n-1)$$
$$\times[\psi(a+n)-\psi(a)-\psi(b+N-n)+\psi(b)], \quad (11)$$

where
$$\psi(x)=\frac{d}{dx}\log\Gamma(x).$$

(9) and (10) are satisfied uniquely by

$$\alpha_n=\frac{kn(b+N-n)}{N(a+b+N-1)}, \quad \beta_n=\frac{k(a+n)(N-n)}{N(a+b+N-1)},$$

which do not satisfy (11). Thus this compound binomial cannot arise from a process of this simple birth and death type.

3. THE MORE GENERAL EQUATIONS

When n is not necessarily bounded by N and α_n and β_n are general functions of time, the equations of the process are

$$\left.\begin{aligned}\frac{dP_n}{dt}&=\alpha_{n+1}(t)P_{n+1}-(\alpha_n(t)+\beta_n(t))P_n+\beta_{n-1}(t)P_{n-1},\\ \frac{dP_0}{dt}&=\alpha_1(t)P_1-\beta_0(t)P_0.\end{aligned}\right\} \quad (12)$$

The first $n+1$ of these add to give

$$\sum_{r=0}^{n}\frac{dP_r}{dt}=\alpha_{n+1}P_{n+1}-\beta_nP_n \quad (n=0,1,\ldots).$$

Thus if it is possible to find a simple closed expression for $\sum_{r=0}^{n}P_r$, we can use this equation to find values for α_n and β_n. Otherwise these values can be found with greater difficulty directly from (12).

3·1. The negative binomial distribution has the form

$$P_n(t)=\frac{\Gamma(k+n)}{\Gamma(k)n!}\frac{\sigma^n}{(1+\sigma)^{n+k}} \quad (n=0,1,\ldots).$$

Suppose k is constant and the distribution is generated from $P_0(0) = 1$, i.e. $\sigma(0) = 0$. Then, if $\sigma' = d\sigma/dt$,

$$\frac{dP_n}{dt} = \frac{\Gamma(k+n)}{\Gamma(k)\,(n-1)!}\frac{\sigma^{n-1}\sigma'}{(1+\sigma)^{n+k}} - \frac{\Gamma(k+n+1)}{\Gamma(k)\,n!}\frac{\sigma^n\sigma'}{(1+\sigma)^{n+k+1}} = f(n-1) - f(n)$$

say, and

$$\sum_0^n \frac{dP_r}{dt} = -P_n\frac{(n+k)\,\sigma'}{1+\sigma} = P_n\left(\alpha_{n+1}\frac{k+n}{n+1}\frac{\sigma}{1+\sigma} - \beta_n\right).$$

The required equation in n is thus, for $n = 0, 1, \ldots$,

$$\frac{-\sigma'}{1+\sigma} = \frac{\alpha_{n+1}}{n+1}\frac{\sigma}{1+\sigma} - \frac{\beta_n}{n+k}. \tag{13}$$

The solution for σ when α_n and β_n are given positive continuous functions is, as required, $\geqslant 0$. The simplest solution is Kendall's (1948*a*), $\alpha_{n+1} = \alpha(n+1)$ and $\beta_n = \beta(n+k)$, where α and β are positive constants. But it is also true that negative binomial will result when α and β vary with time. Generally

$$\sigma = \exp\left(-\int_0^t (\alpha-\beta)\,dt\right)\int_0^t \beta\exp\left(\int_0^t (\alpha-\beta)\,dt\right)dt.$$

Polynomial solutions of higher degrees for α_n and β_n will be more particular solutions in the sense that the coefficients of $n, n^2, \ldots$, cannot be assigned arbitrarily in both α_n and β_n.

3·2. For the positive binomial with probability parameter p and index N, the equation corresponding to (13) is

$$\frac{dp}{dt} = -\frac{\alpha_{n+1}p}{n+1} + \frac{\beta_n(1-p)}{N-n} \quad (n = 0, 1, \ldots, N-1). \tag{14}$$

A solution $\alpha_{n+1} = \alpha(t)\,(n+1)$, $\beta_n = \beta(t)\,(N-n)$ is then possible for positive functions $\alpha(t)$ and $\beta(t)$. For instance, if n describes the number of regions out of N occupied by a species of plants, it is reasonable to suppose that the rate of new colonization is at its highest after each seeding time, when the rate of loss of colonies is at its lowest, i.e. α_n and β_n are periodic and in anti-phase, a situation typified for example by

$$\left.\begin{aligned}\alpha_n(t) &= (\tfrac{1}{2}\omega)\,(1-\sin\omega t)\,n,\\ \beta_n(t) &= (\tfrac{1}{2}\omega)\,(1+\sin\omega t)\,n.\end{aligned}\right\} \tag{15}$$

For these values, $p = \frac{1}{2}\{1 + (1/\sqrt{2})\sin(\omega t - \frac{1}{4}\pi)\}$, having a time lag on $\beta_n(t)$ of half the phase length of α_n and β_n, corresponds to $p(0) = \frac{1}{4}$.

The Poisson equation similar to (13) and (14) is, for a mean m,

$$\frac{dm}{dt} = -\frac{\alpha_{n+1}m}{n+1} + \beta_n,$$

which again is satisfied by arbitrary positive functions $\alpha(t)$ and $\beta(t)$ in $\alpha_n = \alpha(t)\,n$ and $\beta_n = \beta(t)$.

3·3. The geometric distribution is the case $k = 1$ of (13), but the required birth probability rate $\beta_n = \beta(t)\,(n+1)$ is not that usually met in the genesis of a geometric law, viz. $\beta_n = \beta(t)\,n$. The difference arises from the different starting points. To get (13) we took $P_0(0) = 1$, and considered a formula for P_n which held for all values of n including 0. Clearly

a birth-rate proportional to the population size will not move a process away from $P_0(0) = 1$. If, however, $P_1(0) = 1$, a solution for the birth- and death-rates follows from the supposition that P does not necessarily agree with the formula for P_n with $n = 0$.

Then

$$P_n(t) = \frac{1-P_0(t)}{1+\sigma}\frac{\sigma^{n-1}}{(1+\sigma)^{n-1}} \quad (n = 1, 2, \ldots).$$

The equations obtained from addition of the process equations are

$$\frac{\sigma P_0'}{1-P_0}(1+\sigma) - n\sigma' = \alpha_{n+1}\sigma - \beta_n(1+\sigma) \quad (n = 1, 2, \ldots), \tag{16}$$

and the particular equation

$$\frac{dP_0}{dt} = \alpha_1\frac{1-P_0}{1+\sigma} - \beta_0 P_0. \tag{17}$$

The simplest solutions are those given by Kendall (1948*b*), viz. $\alpha_n = \alpha(t)\,n$ and $\beta_n = \beta(t)\,n$. For these, (16) and (17) reduce to

$$\sigma' = -(\alpha-\beta)\sigma + \beta$$

and

$$\frac{P_0'}{1-P_0} = \frac{\alpha}{1+\sigma}, \tag{18}$$

the solutions of which are valid under the initial conditions $P_0(0) = 0$, $\sigma(0) = 0$.

3·4. The logarithmic distribution for the initial conditions $P_1(0) = 1$, $\sigma(0) = 0$, is given by

$$P_n(t) = \frac{1-P_0}{\log(1+\sigma)}\frac{\sigma^n}{n(1+\sigma)^n} \quad (n = 1, 2, \ldots).$$

We require

$$\frac{n\sigma'}{\sigma(1+\sigma)} + \frac{d}{dt}\left[\log\left(\frac{1-P_0}{\log(1+\sigma)}\right)\right] = \frac{\alpha_{n+1}n\sigma}{(n+1)(1+\sigma)} - (\alpha_n+\beta_n) + \frac{\beta_{n-1}n(1+\sigma)}{(n-1)\sigma}, \tag{19}$$

together with two special equations for P_1 and P_0. Suppose polynomial solutions are sought for α_n and β_n. Then (19) when multiplied by $n-1$ becomes a polynomial identity in n. Thus α_{n+1} is divisible by $n+1$, and β_{n-1} by $n-1$. Whence, for $n = 0$,

$$1 - P_0(t) = \text{constant} \times \log(1+\sigma).$$

This is inconsistent with $P_1(0) = 1$, and $\sigma(0) = 0$; thus polynomial solutions are not possible.

But if $P_0(0) = 1$ and $\sigma(0) = 0$, such solutions exist. An artificial one is the disturbed geometric process, $\alpha_n = \alpha n$, $\beta_n = \beta n$ $(n \geqslant 1)$, where α and β are constants with $\alpha > \beta$; and β_0 is a special immigration term to maintain the population if it should drop to zero, viz.

$$\beta_0 = \frac{\beta k}{1 - k\log\left(1 + \frac{\beta}{\alpha-\beta}(1-e^{-(\alpha-\beta)t})\right)}, \quad \text{with} \quad k < \frac{1}{\log\left(\frac{\alpha}{\alpha-\beta}\right)}.$$

Then

$$P_n(t) = \frac{k\sigma^n}{n(1+\sigma)^n} \quad \text{for} \quad n \geqslant 1,$$

$$P_0 = 1 - k\log\left[1 + \frac{\beta}{\alpha-\beta}(1-e^{-(\alpha-\beta)t})\right], \quad \text{where} \quad \sigma = \frac{\beta}{\alpha-\beta}(1-e^{-(\alpha-\beta)t}).$$

When $k \to 0$, $\beta_0 \to 0$, and $P_0(t) \to 1$, but

$$\frac{P_n}{1-P_0} \to \frac{1}{\log(1+\sigma)}\frac{\sigma^n}{n(1+\sigma)^n} \quad \text{for} \quad n \geqslant 1.$$

4. A stochastic development of new species

In the previous paragraphs the discussion has been mainly of the growth of a single species. In this paragraph a simple model for the evolution of a number of different species is considered as a possible source of heterogeneity. Yule (1924) discussed the groupings of species into genera when new species arise by mutation. He assumed that the total number of species in a genus is governed by a geometric birth process, so that the probability that a genus contains n species after a time of establishment t is $e^{-\lambda t}(1-e^{-\lambda t})^{n-1}$, for some constant λ. He remarks that one might also suppose that the probability rate of speciation is dependent on the population sizes of the various species, but gives no guide on the form this dependence might take. One might, as a first approximation, suppose that the larger the population of a species, the greater its chance of producing a member which might be classifiable as a member of a new species. We shall assume that each species develops from its original member with birth and death probability rates proportional to the population size n of the species. The extra assumption is that there is a rate of speciation also proportional to this number. Suppose it is kn. There are difficult, incompletely answered questions as to whether there is an exact time of appearance of a new species, what degree of difference it must show from other existing species, and what the effect is of relative geographical isolation of groups of members of a species. We have had to suppose that a species arises at a definite time. Our reply to the second question is that differences of sufficient size are likely to occur when the population size is big enough. Finally, if isolated groups come together again, it seems plausible that the distinctions necessary to stop interbreeding will have most probably occurred at some past time when the total population size was large; so that again the proportionality of speciation rate to size, and the fixing of a definite time of birth of a species are relevant. Then, too, the use of the geometric law for each species enables us, because of its linear birth and death relationship, to consider the population as a whole even though it is split into groups. It is not pretended, however, that the assumptions made are anything but a rough approximation showing, perhaps, the sort of thing that might happen if more factors were taken into account. Thus because of the other factors involved, such as the struggle for a changing amount of food of varying quality, the constant k must be regarded as summing up the mutation rates and the environmental variability.

We shall be concerned only with the number of species and not their genera or higher groupings.

4·1. The distribution of the number of species that have been born in $(0, t)$ is not easily obtainable. Suppose N species have been created at times $\tau_1, \tau_2, \ldots, \tau_N$, and that initially there was one species with one member. Then, if $g_n(t)$ is the probability that a species has n members after time t, the probability of a further birth in $(t, t+dt)$ is

$$k \sum_{i=0}^{N} \sum_{n=0}^{\infty} n(t-\tau_i)\, g_n(t-\tau_i) = k \sum_{i=0}^{N} \bar{n}(t-\tau_i), \quad \text{say.}$$

Suppose
$$k \sum_{i=0}^{j-1} \bar{n}(\tau_j - \tau_i) = \alpha_j(\tau_0, \ldots, \tau_j) \quad \text{with} \quad \tau_0 = 0,$$

and
$$p(0, \tau) = \exp\left(-\int_0^{\tau} \bar{n}(x)\, dx\right).$$

Then it can be shown that $P_{N+1}(t)$, the probability that there are $N+1$ species at time t, is

$$\int_{R_N} p(0,t)\,p(0,t-\tau_1)\dots p(0,t-\tau_N)\prod_{j=1}^{N}\alpha_j(\tau_0,\dots,\tau_j)\,d\tau_1\dots d\tau_N$$
$$=\int_{R_N} P(\tau_1,\dots,\tau_N,t)\,d\tau_1\dots d\tau_N, \quad \text{say,} \tag{20}$$

where R_N is the region given by the inequalities $0\leqslant\tau_1\leqslant\tau_2,\ 0\leqslant\tau_2\leqslant\tau_3,\dots,0\leqslant\tau_N\leqslant t$. $P_{N+1}(t)$ then follows a birth process with transition rate $\beta_{N+1}(t)$ given by

$$\beta_{N+1}(t)\,P_{N+1}(t)=\int_{R_N} P(\tau_1,\dots,\tau_N,t)\,\alpha_{N+1}(\tau_0,\dots,\tau_N,t)\,d\tau_1\dots d\tau_N. \tag{21}$$

It is hardly possible to use these results except for the simplest form of geometric distribution. It would be most satisfactory to deduce a multivariate distribution of the different numbers of species which at time t have $0,1,\dots$ members. Failing this we can find the average number of species having a given population. Since N follows a birth process, we may write the equation for the mean number, $\overline{N}$, of species at time t,

$$\frac{d\overline{N}}{dt}=\sum_N \beta_N P_N.$$

Then the equation for the expected number N_n of species with n members is

$$N_n=\int_0^t \Sigma\beta_N P_N g_N(t-\tau)\,d\tau+g_n(t)$$
$$=\int_0^t \frac{d\overline{N}}{d\tau}\,g_n(t-\tau)\,d\tau+g_n(t). \tag{22}$$

When this is multiplied by n and summed, our initial assumption gives

$$\frac{d\overline{N}}{dt}=k\left[\int_0^t \frac{d\overline{N}}{d\tau}\,\overline{n}(t-\tau)\,d\tau+\overline{n}(t)\right]. \tag{23}$$

This has the unique solution for the Laplace transforms ϕ of $\overline{n}$, and ψ of $d\overline{N}/dt$

$$\psi=\frac{k\phi}{1-k\phi}. \tag{24}$$

4·2. The simplest example occurs when both birth- and death-rates λ and μ are equal to λ. Then

$$g_n(t)=\frac{(\lambda t)^{n-1}}{(1+\lambda t)^{n+1}}\quad (n=1,2,\dots),\qquad g_0(t)=\frac{\lambda t}{1+\lambda t},\quad \text{with}\quad g_1(0)=1.$$

This is perhaps the most realistic of the geometric laws, since it corresponds to a group of species so near, on the average, to equilibrium with their environment, that a new species has difficulty in establishing itself. R. A. Fisher (1930) holds that it is reasonable to suppose that most mutant genes that are likely to assist in the development of a species will probably disappear from the population in time. For instance, a gene giving a 1 % advantage will die out with probability 0·98, while one giving no advantage will die out with probability 1. It is commonly considered that most mutants that can survive make little effective change in an organism's relationship to its surroundings. Thus in the present artificial situation in which a new species is supposed to be created at a definite time, it must be assumed that it is not much better adapted than the others. If, as postulated, a species grows geometrically, its chance of possessing no members after time $t\to\mu/\lambda$ if $\lambda>\mu$, and to 1 otherwise. Amongst such laws of growth then, the one most representative of what happens is that for which $\lambda=\mu$. Then $\overline{n}(t)=1$, and $P_N(t)=e^{-kt}(1-e^{-kt})^{N-1}$ as in

Yule's case, $\overline{N} = e^{kt}$. When the distribution (22) is applied to actual data, the values of kt and λt required are such that the distribution is very near its limiting form

$$\left.\begin{aligned} p_n &= \lim_{t\to\infty} \frac{N_n}{\overline{N}} = \int_0^\infty \frac{\alpha e^{-\alpha y} y^{n-1}}{(1+y)^{n+1}}\, dy \quad (n = 1, 2, \ldots), \\ p_0 &= 1 - \int_0^\infty \frac{\alpha e^{-\alpha y}}{1+y}\, dy, \quad \text{where} \quad \alpha = k/\lambda. \end{aligned}\right\} \tag{25}$$

The p_n depend only on the ratio of k to λ, since these both have the dimension 1/time. The following properties hold:

(i) $p_n > p_{n+1}$ for all $\alpha > 0$, and $n \geqslant 1$.

(ii) p_0 may range from 0 as $\alpha \to \infty$, to 1 as $\alpha \to 0$. As $\alpha \to \infty$, $p_1 \to 1$.

(iii) The rth factorial moment of (25) is $r!(r-1)!/\alpha^{r-1}$. Hence for small k, despite the fact that $p_0(\alpha) \to 1$, the moments become unbounded. This is implied in the increasing average age, $1/\alpha$, of a species, since, as a species grows older, g_n still increases if $n - 1 > 2\lambda \times$ its age, i.e. the larger contributions to the moments increase. When α is large, the ordinary moments all $\to 1$. The overwhelming majority of species are only recently born, and have in virtue of the initial condition for each species only about one member each.

4·3. *Variation in* α. One of the chief defects of the model is the assumption that all species have the same k and λ. In (24) k can be looked on as a mean value, if it is assumed to be independent of time, since it occurs linearly. λ does not occur till (25). In (25) then, it is possible to allow $\alpha = k/\lambda$ to have its own distribution. Because of (i) the resulting distribution of n will have the same general shape as (25).

For (25) there is the recurrence relationship

$$(n+1)\, p_{n+1} = n p_n + \alpha(p_1 + \ldots + p_n) + \alpha^2 e^\alpha \operatorname{Ei}(-\alpha) \quad (n = 1, 2, \ldots), \tag{26}$$

where $$-\operatorname{Ei}(-\alpha) = \int_\alpha^\infty \frac{e^{-x}}{x}\, dx \quad (\textit{British Association Tables}, \text{ vol. 1})$$

and $$p_1 = \alpha(1 + \alpha e^\alpha \operatorname{Ei}(-\alpha)) = \alpha(1 - p_0).$$

The terms extra to np on the right-hand side are of order $\alpha + \alpha^2 e^\alpha \log \alpha$ for small α. The mean value of α will be small, since an ordinary birth is much more common than a species-generating birth. Hence one expects that $E_\alpha p_n$ will vary only a little from a logarithmic form, constant $\times x^n/n$ for small $1 - x$. But a small α implies by (iii) a distribution with very large moments. A distribution that leaves all moments of $E_\alpha p_n$ finite does not lead to easily calculable probabilities. The most obvious $dF(\alpha)$ is $dF(\alpha) = (c^m/\Gamma(m))\, e^{-c\alpha} \alpha^{m-1} d\alpha$. Then $[m+1]$ moments are finite. When c is large, as is required by fitted data, the new probabilities may most easily be calculated from

$$\left.\begin{aligned} p_n &= \frac{m}{cn} - \frac{m(m+1)}{c^2}\left[\log c - \psi(n+1) - \psi(m+2) + \psi(1) + \psi(2) + \ldots \right.\right. \\ &\quad + \frac{(n+1)\ldots(n+r)\,(m+2)\ldots(m+r+1)}{r!(r+1)!\, c^r}(\log c - \psi(n+r+1) \\ &\quad \left.\left. - \psi(m+r+2) + \psi(r+1) + \psi(r+2)) + \ldots\right], \\ 1 - p_0 &= \frac{m(m+2)}{c(m+1)} \log c\left(\frac{c}{c-1}\right)^{m+1} + \frac{m(m+2)}{c(m+1)} \\ &\quad \times \left[\psi(1) - \psi(m+1) + \ldots + \frac{(m+1)\ldots(m+r)}{r!\, c^r}(\psi(r+1) - \psi(m+r+1)) + \ldots\right]. \end{aligned}\right\} \tag{27}$$

(27) can be fitted from the first two moments. Thus if u is the observed mean, and v is the ratio of the observed second factorial moment and the mean, the equations are, since the zero class is not observable,

$$\frac{2c}{m-1} = v \tag{28}$$

and

$$\frac{1}{u} = 1 - p_0 = -\alpha e^{\alpha} \operatorname{Ei}(-\alpha). \tag{29}$$

(28) gives c in terms of m; there will only be a solution for m in (29) if

$$\frac{1}{u} > \frac{2}{v} e^{2/v} (-\operatorname{Ei}(-2/v)).$$

However, in the case tried, namely C. B. Williams's data on different species of Macrolepidoptera caught in light traps over 4 years (Fisher, Corbet & Williams, 1943), a solution is not possible. The nearest approach is when c and $m \to \infty$ in the limiting ratio $2/v$, i.e. when the distribution is back to the form (25). Then a better fit is obtained from the same distribution with the value of α given by the equation $1/u = -\alpha e^{\alpha} \operatorname{Ei}(-\alpha)$.

Generally, for *any* distribution of α, it is to be expected that the maximum-likelihood fitting of $E_{\alpha} p_n$ will be very difficult. If, then, one were to fit by moments, including the second, the equation $E_{\alpha}(2/\alpha) = v$ would be used. Since $E(2/\alpha) \geqslant 2/E(\alpha)$, the smallest mean of $dF(\alpha)$ would be $2/v$, which is reached only when $dF(\alpha)$ tends to the form $P(\alpha \neq \alpha_0) = 0$, $P(\alpha_0) = 1$. For the sort of distribution $dF(\alpha)$ we would be likely to consider, this would mean that the variance of α would $\to 0$ as $E(\alpha) \to 1/E(1/\alpha) = 2/v$. The data, of typical 'log' shape, are fairly well fitted by (25) with $\alpha = 0{\cdot}00290$ and $2/v = 0{\cdot}00326$. Thus it seems likely that a $dF(\alpha)$ leading to a better fit will have to have a mean near 0·00290 and so tending to 0·00326 from above. Thus it will have to have a low variance and there will be little point in using something like (27) rather than (25). An improvement might be found in allowing λ to be different from μ in the geometric law $g_n(t)$.

4·4. *The average distribution with unequal λ and μ.* When $\lambda > \mu$ the limiting distribution of the type (22) has infinite moments of all but a few orders, and is thus less satisfactory than (25) as a description of data of this kind.

When $\mu > \lambda$ the distribution is

$$p_n = \frac{(\lambda - \mu + k)(\mu - \lambda)}{\mu^2} \left(\frac{\lambda}{\mu}\right)^{n-1} \int_0^1 x^{-1+k/(\mu-\lambda)} (1-x)^{n-1} \left(1 - \frac{\lambda}{\mu} x\right)^{-(n+1)} dx,$$

in which $k - (\mu - \lambda)$ must be > 0, as the average distribution only makes sense if a large number of species has been born ($\bar{N}$ the average number of species born up to time t is proportional to $\exp(k - (\mu - \lambda) t)$.

If $\xi = k/(\mu - \lambda)$, p_n may be written as

$$\frac{(\xi - 1)}{(\mu/(\mu - \lambda))^2} \left(\frac{\lambda}{\mu}\right)^{n-1} \frac{\Gamma(\xi)\,\Gamma(n)}{\Gamma(n+\xi)} F(n+1, \xi; n+\xi; \lambda/\mu) \quad (n = 1, 2, \ldots).$$

If λ/μ is nearly 1, $= \eta/(\eta+1)$ say, the easiest form for calculation for p_0 is, when η is large,

$$\begin{aligned} 1 - p_0 &= \frac{\xi - 1}{\eta + 1} \int_0^1 x^{\xi - 1} (1 - (\lambda/\mu) x)^{-1} dx \\ &= \frac{(\xi - 1)(\eta + 1)^{\xi - 1}}{\eta^{\xi}} \log(\eta + 1) - \sum_{r=0}^{\infty} \frac{(\xi - 1)\,\xi \ldots (\xi + r - 1)}{r!} \frac{\psi(\xi + r) - \psi(r + 1)}{(\eta + 1)^{r+1}}. \end{aligned} \tag{30}$$

Then later p_n's can be calculated from the recurrence relationship

$$(n+1)p_n = n\frac{\lambda}{\mu}p_n + \frac{\xi-1}{\eta+1}(p_1+\ldots+p_n) - \frac{\xi-1}{\eta+1}; \quad \left(p_1 = \frac{\xi-1}{\eta+1}p_0\right); \quad (n = 1, 2, \ldots). \tag{31}$$

The chief difference between (31) and (27) is that in (31) the probabilities are exhibited as corrections to a logarithmic term, whereas in (27) they are given as corrections to a harmonic term. Both correction terms are small. The moments of the distribution determined by (31) are all finite. One may find ξ and η from the equations

$$\left.\begin{aligned} \frac{2\eta}{1+\xi} &= v, \\ \xi\int_0^1 \frac{x^{\xi-1}}{1+\eta(1-x)}\,dx &= \frac{1}{u}. \end{aligned}\right\} \tag{32}$$

The values of the right-hand side range from $\log(1+v)/v$ when $\xi = 1$, down to

$$(2/v)\exp(2/v)\,(-\mathrm{Ei}(-2/v))$$

as $\xi \to \infty$ (when η is substituted in terms of ξ). For these data $\xi = 7{\cdot}3$, $\eta = 2549{\cdot}0$. The first 25 frequencies and groupings up to an beyond 50 are compared with those for the log distribution. They agree closely after the first. The figures for the α distribution are so close to those for the ξ, η distribution as not to be worth recording separately. When α is small, as here ($\alpha = 0{\cdot}00290$), it is approximately $1-x$ when the log distribution is given by $N_n = \text{const.}\, x^n/n$. The value of η shows that if this model is at all relevant, the value of λ/μ is almost indistinguishable from 1. The interesting suggestion from the ratio ξ/η is that one species-generating birth occurs about every 350 generations.

Macrolepidoptera figures for all four years

n	N_n obs.	N_n ξ, η distribution	N_n log distribution $x = 0{\cdot}99742$	n	N_n obs.	N_n ξ, η distribution	N_n log distribution $x = 0{\cdot}99742$
1	35	44·09	40·14	17	3	2·25	2·27
2	11	21·80	20·03	18	3	2·11	2·14
3	15	14·38	13·32	19	3	1·98	2·02
4	14	10·54	9·96	20	4	1·87	1·92
5	10	8·36	7·95	21	1	1·77	1·82
6	11	6·90	6·66	22	3	1·68	1·73
7	5	5·87	5·65	23	3	1·60	1·65
8	6	5·09	4·93	24	1	1·52	1·58
9	4	4·49	4·37	25	3	1·45	1·51
10	4	4·01	3·92	26–30	7	6·40	6·71
11	2	3·62	3·56	31–35	3	5·27	5·62
12	2	3·29	3·25	36–40	5	4·46	4·81
13	5	3·02	3·00	41–45	5	3·84	4·19
14	2	2·78	2·77	46–50	3	3·36	3·71
15	4	2·58	2·58	Over 50	60	57·22	63·81
16	3	2·40	2·42				
				Total	240		

4·5. *Use of the model for a finite distribution.* Suppose as in (1) that species of plants are spreading over M regions. Then the spread may possibly be described by some simple birth and death process (α_n, β_n), with α_n and β_n not depending on time; we might, for instance, consider the logistic process $\alpha_n = \mu n^2$, $\beta_n = \lambda n(M-n)$, where n is the number of regions occupied by a species. If one supposes that the mean number N of species increases exponentially as in 4·2 and 4·4, the average rate of production of new species is

$$\sum_N \beta_N(t) P_N(t) = a e^{at},$$

where a is a constant. Then the age distribution of the $\overline{N} = e^{at}$ species $\to a e^{-ax} dx$ as $t \to \infty$. Suppose the probability that a species of age x occupies n regions is $g_n(x)$ when $g_1(0) = 1$. The limiting distribution p_n that we have been using can then be obtained directly from the equations of the process for the $g_n(x)$. Thus

$$\begin{aligned} a\int_0^\infty e^{-at}\frac{dP_n(t)}{dt}dt &= a^2\int_0^\infty e^{-at}P_n(t)\,dt - a_n P_n(0) \\ &= \alpha_{n+1}a\int_0^\infty e^{-at}P_{n+1}dt - (\alpha_n+\beta_n)a\int_0^\infty e^{-at}P_n dt \\ &\quad + \beta_{n-1}a\int_0^\infty e^{-at}P_{n-1}dt \quad (N \geqslant n > 1), \end{aligned} \tag{33}$$

or, as

$$a\int_0^\infty e^{-at}P_n dt = p_n,$$

$$ap_n = \alpha_{n+1}p_{n+1} - (\alpha_n+\beta_n)p_n + \beta_{n-1}p_{n-1}, \tag{34}$$

with the special end equations,

$$-a + ap_1 = \alpha_2 p_2 - (\alpha_1+\beta_1)p_1 + \beta_0 p_0,$$

$$ap_0 = \alpha_1 p_1 - \beta_0 p_0,$$

and

$$ap_N = -\alpha_N p_N + \beta_{N-1}p_{N-1}.$$

The equations add to give a recurrence relationship such as (26) or (31),

$$\alpha_{n+1}p_{n+1} = \beta_n p_n + a(p_0 + \ldots + p_n) - a \quad (n = 1, 2, \ldots). \tag{35}$$

When p_0 can be found, as is the case in 4·2, (35) can be used to give the p_n's. For $n \leqslant M$, p_n is of the form $A_n p_0 + B_n$, the coefficients A_n and B_n being obtained from (35). Then p_n can be found by addition of the probabilities.

To show the possibility of such 'explanation' of the distribution of species, we can fit data (kindly supplied by Prof. A. R. Gemmell) on the spread of moss species over the six islands of Hawaii. The process $\alpha_n = 0$, $\beta_n = \lambda n(M-n)$ has been used. This is not quite unrealistic, as a species is only observed when it has a large number of plants and so is not so likely to disappear. However, even with $\alpha_n = 0$, the fits are good:

$$\left.\begin{aligned} p_n &= \frac{(n-1)(M-(n-1))}{\gamma+n(M-n)}\,\frac{(n-2)(M-(n-2))}{\gamma+(n-1)(M-(n-1))}\cdots\frac{1(M-1)}{\gamma+2(M-2)}\,\frac{\gamma}{\gamma+1(M-1)} \\ p_1 &= \frac{\gamma}{\gamma+1(M-1)}, \quad \gamma = k/\lambda. \end{aligned}\right\} \tag{36}$$

The figures are, for the three different reproductive types of species, dioecious, monoecious and sterile:

Moss species in Hawaii

Dioecious			Monoecious			Sterile		
n	Obs.	Exp.	n	Obs.	Exp.	n	Obs.	Exp.
1	12	11·19	1	15	13·88	1	22	21·78
2	6	5·70	2	6	5·64	2	7	7·19
3	4	4·22	3	3	3·39	3	5	3·56
4	3	3·87	4	2	2·48	4	1	2·11
5	4	4·53	5	0	2·13	5	0	1·39
6	13	12·49	6	4	2·47	6	2	0·97
$\gamma = 1{\cdot}816$			$\gamma = 4{\cdot}308$			$\gamma = 7{\cdot}156$		

γ was calculated in each case from the maximum-likelihood equation

$$\sum_n N_n\left(\frac{1}{\gamma+n(M-n)}+\ldots+\frac{1}{\gamma+1(M-1)}\right)=(\sum_n N_n)\frac{1}{\gamma}.$$

4·6. *Remarks.* The simple assumption of proportionality of speciation rate to the total population thus leads to distributions substantially like those observed. Yule's data for the increasing number of species in a genus bears out the exponential manner of increase found for N. For such an N the form of the p_n can always be found as in 4·5, and more realistic growth laws for the different species may account for any divergences in observed data from a logarithmic form. For instance, random immigration of new species or of members of existing species (which might include back mutation to other already existing species) may be considered. But these particular effects are of small weight in the p_n unless their rate of occurrence is of as high an order as the rate of increase of N. A more important change might be the consideration of competitive populations described by something like the logistic law, but they are possibly best considered in relation to the shifting of the population inside the living region.

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MODIFICATIONS TO THE VARIATE-DIFFERENCE METHOD

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1. Introduction

1·1. Methods of trend elimination in time-series analysis fall into two main groups. First, where the form of trend is known or may be assumed to be polynomial, it is possible to fit polynomials and to use residuals from this fitting in further analysis. This method has been described elsewhere (Quenouille, 1952).

A second, more exploratory, method which assumes nothing about the form of the trend makes use of moving average difference formulae for its estimation. This method is, however, subject to the criticisms that the form and extent of moving averages chosen would seem to be somewhat arbitrary, and that the elimination process may easily give rise to misleading results (Spencer-Smith, 1947; Quenouille, 1949).

1·2. The variate-difference method (Anderson, 1914) is closely related to the method of moving averages, and may be regarded as having greater generality. Any moving average difference formula may be shown to be equivalent to an average of higher order differences and, consequently, results obtainable from such formulae may be obtained by appropriate manipulation of the variate-differences. For example, if $\nabla^p x_i$ is used to denote the pth variate difference, i.e.

$$\nabla^p x_i = x_{i+p} - \binom{p}{1} x_{i+p-1} + \binom{p}{2} x_{i+p-2} - \ldots,$$

then a simple five-term moving average difference formula may be written

$$\begin{aligned} 5d_5 &= -x_{i+2} - x_{i+1} + 4x_i - x_{i-1} - x_{i-2} \\ &= -\nabla^2 x_i - 3\nabla^2 x_{i-1} - \nabla^2 x_{i-2}. \end{aligned}$$

Results obtainable using the residuals from such a formula are therefore derivable from the variate-differences.

1·3. The classical variate-difference method has therefore a greater flexibility than the use of moving averages, since the effect of the latter may be judged from the variate-difference. Further, it has the advantage that criteria and tests of significance can be set up to judge the extent of the trend.

A major disadvantage still exists in that the existence of serial correlation in the residual element of the series will bias the results and produce misleading conclusions about the form of the series. Also, it is not clear that the tests used when serial correlation is not present in the residual element are the best possible.

1·4. The following paper sets out to do two things. The first part, §§ 2–6, is concerned with deriving more sensitive tests of the existence of trend in variate-differences of any order, and with finding more accurate estimates of variances and covariances of the residual element from any set of variate-differences assumed to be trend free. The latter part, §§ 7–10, is concerned with how this theory should be modified if serial correlation is assumed to exist in the residuals. As such it does not give a strict criterion for the separation of trend and serial correlation, but it does provide a basis on which the separation of the two may take place.

For example, the analysis of §7·3 indicates the existence of both trend and serial correlation in Kendall's sheep series. It shows further that trend is unlikely to be important in the third and higher variate-differences. The analysis of §9·5 shows further that, *if* it is accepted that there is no trend in the third and higher variate-differences, at least a second-order autoregressive process is needed to represent the series and that, *if* such is used, the variance of the trend in the second variate-differences is unlikely to be more than 0·4. The conclusion is therefore that a second-order autoregressive scheme with a trend, small in its second differences, is a good representation of the series. As emphasized elsewhere (Quenouille, 1949), this is not, however, the only possible representation, and it would still be possible to maintain that the series consisted of a complicated trend with a random element or, possibly, of a high-order autoregressive scheme with no trend. The process to be outlined gives rise to the simplest possible representation for any time series.

2. General investigation

2·1. The classical variate-difference method is based upon the idea that by taking successive differences of a series of observations, the 'smooth' part or trend of the observations will be eliminated, and what remains can be used to find out about the random element. In particular, if the series $x_1, x_2, \ldots, x_{N+p}$ is free from serial correlation and no trend component exists in the pth variate-difference, $\nabla^p x_i$, then

$$\frac{(\nabla^p x_i)^2}{\binom{2p}{p}}$$

estimates the variance, σ^2, of the random element.

Any average of a series of such terms also estimates σ^2, and, commonly, estimates such as

$$V_p = \frac{\sum_{i=1}^{N} (\nabla^p x_i)^2}{N\binom{2p}{p}}$$

are used for this purpose.

For reasons to be seen later, we shall use a modified definition which applies *end-corrections* to the averaging process.

This will not affect the large sample theory, but it will have the joint effects of making it more applicable to small samples and of simplifying conversion formulae.

2·2. We shall define a series of summation functions $\Sigma_{(l)}$ by the formulae:

$$\begin{aligned}
\Sigma_{(0)} x_i y_i &= x_1 y_1 + x_2 y_2 + \ldots + x_n y_n, \\
\Sigma_{(1)} x_i y_i &= \tfrac{1}{2} x_1 y_1 + x_2 y_2 + \ldots + x_{n-1} y_{n-1} + \tfrac{1}{2} x_n y_n, \\
\Sigma_{(2)} x_i y_i &= \tfrac{1}{4} x_1 y_1 + \tfrac{3}{4} x_2 y_2 + \ldots + \tfrac{3}{4} x_{n-1} y_{n-1} + \tfrac{1}{4} x_n y_n,
\end{aligned}$$

the summation in each case covering all values up to x_n, y_n. The general law of formation is determined by the recurrence relation

$$\sum_{i=1}^{n}{}_{(m)} x_i y_i = \sum_{i=1}^{n-1}{}_{(m-1)} \tfrac{1}{2}(x_i y_i + x_{i+1} y_{i+1}),$$

so that, for example, the first three coefficients in $\Sigma_{(3)}$ are $\frac{1}{8}$, $\frac{1}{2}$ and $\frac{7}{8}$.

2·3. For any series, $x_1, x_2, \ldots, x_m$, the estimated variances, ${}_mV_0, {}_mV_1, {}_mV_2, \ldots$, will be defined by

$$
\begin{aligned}
{}_mV_0 &= \frac{\Sigma_{(m)}x^2}{N}, \\
{}_mV_1 &= \frac{\Sigma_{(m-1)}(\nabla x)^2}{2N}, \\
&\vdots \qquad\qquad \vdots \\
{}_mV_p &= \frac{\Sigma_{(m-p)}(\nabla^p x)^2}{N\binom{2p}{p}}.
\end{aligned}
$$

Under this definition, p cannot exceed m. However, m may be arbitrarily chosen so that it can always be arranged that it is equal to or greater than the largest value of p required.

In the following sections, m will be taken as equal to the largest value of p occurring. This is usually ten. Further, the prefix m will, in general, be omitted unless the line of argument requires its inclusion.

2·4. Serial correlation coefficients may be similarly defined, so that

$${}_mr_p = \frac{\Sigma_{(m-p)}x_i x_{i+p}}{\Sigma_{(m)}x_i^2}.$$

Again, m must be chosen to equal or exceed p, and, with this condition, the suffix m may usually be omitted.

2·5. It is then not very difficult to derive the following identities (see Kendall, 1946, p. 23):

$$V_p = V_0\left[1 - \frac{2p}{p+1}r_1 + \frac{2p(p-1)}{(p+1)(p+2)}r_2 - \frac{2p(p-1)(p-2)}{(p+1)(p+2)(p+3)}r_3 - \ldots\right]$$

and

$$r_pV_0 = V_0 - p^2V_1 + \frac{p^2(p^2-1)}{(2!)^2}V_2 - \frac{p^2(p^2-1)(p^2-4)}{(3!)^2}V_3 + \ldots.$$

These formulae demonstrate the basic equivalence of the approaches employing variate-differences and serial correlations. They also provide a simple way in which to find the variances of combinations of the V_i under different conditions. For instance,

$$\operatorname{var} V_p = \frac{2\sigma^4}{N}\left[1 + \frac{2p^2}{(p+1)^2} + \frac{2p^2(p-1)^2}{(p+1)^2(p+2)^2} + \ldots\right]$$

for a random series.

2·6. When a set of variate-differences have been calculated as described above, it is reasonable to ask what is the best estimate of the residual or error variance in any series that may be obtained using these estimates. If it is supposed that $V_{m+1} \ldots V_{m+p}$ are free from any trend component, then the problem is to find coefficients c_i $(i = m+1 \ldots m+p)$ so that $\sum_{i=m+1}^{m+p} c_i = 1$ and so that $\sum_{i=m+1}^{m+p} c_iV_i$ has the smallest possible variance.

This problem may be solved by finding a restricted minimum. If we note that

$$\frac{N}{\sigma^4}\left[\operatorname{cov}(V_i, V_j) - \frac{\kappa_4}{N}\right] = \frac{2\binom{2i+2j}{i+j}}{\binom{2i}{i}\binom{2j}{j}} = a_{ij}, \text{ say}$$

where κ_4 is the fourth cumulant of the residual elements, and will normally be assumed to be

zero, and $(\kappa_4+\lambda\sigma^4)/N$ is supposed to be the variance of $\sum_{i=m+1}^{m+p} c_i V_i$, the coefficients c_i are given by the symmetrical equations

$$\left.\begin{aligned} \sum_{i=m+1}^{m+p} a_{ji}c_i - \lambda &= 0 \\ -\sum_{i=m+1}^{m+p} c_i &= -1 \end{aligned}\right\} \quad (j=m+1,\ldots,m+p).$$

Solution of these equations gives rise to the values for c_i and λ.

2·7. The following particular solutions exist for these equations

m	$p=1$		$p=2$		
	c_{m+1}	λ	c_{m+1}	c_{m+2}	λ
0	1	3·00	2·50	−1·50	2·50
1	1	3·89	3·86	−2·86	3·00
2	1	4·62	5·20	−4·20	3·43
3	1	5·25	6·54	−5·54	3·82
4	1	5·82	7·88	−6·88	4·17
5	1	6·33	9·21	−8·21	4·49
6	1	6·81	10·55	−9·55	4·79

m	$p=3$				$p=4$				
	c_{m+1}	c_{m+2}	c_{m+3}	λ	c_{m+1}	c_{m+2}	c_{m+3}	c_{m+4}	λ
0	4·67	− 7·00	3·33	2·33	7·50	− 20·25	22·50	− 8·75	2·25
1	10·36	− 17·91	8·55	2·69	22·71	− 67·28	72·25	− 26·68	2·52
2	18·18	− 33·04	15·86	3·00	50·37	− 152·56	161·08	− 57·90	2·77
3	28·14	− 52·43	25·29	3·28	94·33	− 287·71	300·79	− 106·42	3·00
4	40·23	− 76·09	36·86	3·55	159·00	− 486·23	505·16	− 176·92	3·21
5	54·45	− 104·01	50·56	3·79	245·2	− 749·6	774·4	− 269·0	3·41
6	70·79	− 136·17	66·38	4·02	356·6	− 1088·9	1119·9	− 386·7	3·60

It may be seen that fairly substantial improvements in accuracy can be achieved by using several values of V_i simultaneously. For instance, the variance of V_4 is 5·25. By combining V_4 and V_5 the variance is reduced to 3·82, while the use of V_4 to V_7 reduces it to 3·00.

2·8. It should be noted that the coefficients, c_i, do not differ appreciably from those that would be used if the a_{ij} were regarded as functions of i for which the value corresponding to $i=-1$ was required. For instance, in this case, for $p=2$, we should have $c_{m+1}=m+3$, $c_{m+2}=-(m+2)$; for $p=3$,

$$c_{m+1}=(m+3)(m+4)/2, \quad c_{m+2}=-(m+2)(m+4), \quad c_{m+3}=(m+2)(m+3)/2.$$

This provides rough empirical approximations to the coefficients, c_i, but for practical purposes it is convenient to deal with a still rougher approximation, derived by extrapolating the value of V_i corresponding to $i=0$, using the formula

$$V_0 = V_m - m\nabla V_m + \frac{m(m+1)}{2!}\nabla^2 V_m - \ldots.$$

If this is used, the following coefficients are obtained:

m	$p=1$		$p=2$			$p=3$				$p=4$				
	c_{m+1}	λ	c_{m+1}	c_{m+2}	λ	c_{m+1}	c_{m+2}	c_{m+3}	λ	c_{m+1}	c_{m+2}	c_{m+3}	c_{m+4}	λ
0	1	3·00	2	−1	2·56	3	−3	1	2·42	4	−6	4	−1	2·35
1	1	3·89	3	−2	3·08	5	−6	2	2·89	8	−15	11	−3	2·75
2	1	4·62	4	−3	3·53	10	−15	6	3·17	20	−45	36	−10	2·98
3	1	5·25	5	−4	3·93	15	−24	10	3·48	35	−84	70	−20	3·24
4	1	5·82	6	−5	4·29	21	−35	15	3·76	56	−140	120	−35	3·48
5	1	6·33	7	−6	4·62	28	−48	21	4·03	84	−216	189	−56	3·71
6	1	6·81	8	−7	4·93	36	−63	28	4·28	120	−315	280	−84	3·92

2·9. This table shows that an appreciable increase in the efficiency of estimation of the residual variance is obtained even with simple extrapolation. The efficiency of this extrapolation method as compared with the most efficient linear combinations, obtained in §2·7, is as follows:

m	$p=1$	$p=2$	$p=3$	$p=4$
0	100%	98%	96%	96%
1	100%	97%	93%	92%
2	100%	97%	95%	93%
3	100%	97%	94%	93%
4	100%	97%	94%	92%
5	100%	97%	94%	92%
6	100%	97%	94%	92%

The loss in efficiency increases with both m and p. However, in view of the computational convenience and other properties, to be given later, of the extrapolation method, its use is generally to be advocated. Any loss in efficiency from its use can be compensated by employing more values of V_i.

2·10. If it may be presumed that $V_{m+1} \dots V_{m+p}$ are free from trend, the trend component in V_m may be estimated by comparing it with $V_{m+1} \dots V_{m+p}$. A further question is then posed: which estimate of the trend component in V_m is the most accurate? To answer this, we have to find coefficients $d_i (i = m+1, \dots, m+p)$, subject to $\sum_{i=m+1}^{m+p} d_i = 1$, such that the variance, $(\kappa_4 + \lambda\sigma^4)/N$, of $V_m - \sum_{i=m+1}^{m+p} d_i V_i$, is a minimum. These coefficients may be shown to satisfy the equations

$$\left.\begin{aligned} \sum_{i=m+1}^{m+p} (a_{ji} - a_{mi})\, d_i - \lambda &= a_{jm} - a_{mm} \\ -\sum_{i=m+1}^{m+p} d_i &= -1. \end{aligned}\right\} \quad (j = m+1, \dots, m+p)$$

Solution of these equations gives rise to values for d_i and λ.

2·11. The following particular solutions exist for these equations:

m	$p=1$		$p=2$			$p=3$			
	d_{m+1}	λ	d_{m+1}	d_{m+2}	λ	d_m	d_{m+1}	d_{m+2}	λ
1	1	0·222222	2·327	−1·327	0·030612	4·081	−5·388	2·306	0·007752
2	1	0·108889	2·230	−1·230	0·006949	3·763	−4·636	1·873	0·000924
3	1	0·067347	2·177	−1·177	0·002469	3·587	−4·239	1·652	0·000202
4	1	0·046838	2·143	−1·143	0·001113	3·475	−3·992	1·517	0·000063
5	1	0·034974	2·121	−1·121	0·000582	3·402	−3·834	1·432	0·000022
6	1	0·027391	2·104	−1·104	0·000337	3·345	−3·713	1·368	0·000010

It may be seen that these coefficients tend to binomial coefficients, i.e. $d_{m+i} \doteqdot (-1)^{i-1}\binom{p}{i}$, and this may be proved mathematically without much difficulty. (In the limit, the coefficients $a_{ji}-a_{mi}$ may be accurately represented as polynomials in j. It follows that a limiting solution is given by the differencing process, $d_{m+i} = (-1)^{i-1}\binom{p}{i}$ and $\lambda = 0$.) Estimates of trend components are thus given by variate-differences of the V_i.

2·12. The function $\sum_{i=m}^{m+p}(-1)^{m+i}\binom{p}{i-m}V_i$ thus estimates the trend in V_m, if $V_{m+1}\dots V_{m+p}$ are free from trend. Its variance is given by

$$\frac{2\sigma^4}{N}\sum_{i=0}^{p}\sum_{j=0}^{p}(-1)^{i+j}\frac{\binom{p}{i}\binom{p}{j}\binom{4m+2i+2j}{2m+i+j}}{\binom{2m+2i}{m+i}\binom{2m+2j}{m+j}} = \frac{\lambda\sigma^4}{N}.$$

This is of order m^{1-2p}. The following table gives the values of λ:

m	Values of p					
	0	1	2	3	4	5
1	3·0000	0·2222	0·042222	0·015918	0·007927	0·004602
2	3·8889	0·1089	0·010522	0·002394	0·000796	0·000330
3	4·6200	0·0673	0·003935	0·000599	0·000142	0·000043
4	5·2531	0·0468	0·001833	0·000200	0·000035	0·000007
5	5·8187	0·0350	0·000980	0·000080	0·000011	0·000002
6	6·3346	0·0274	0·000577	0·000037	0·000004	—
7	6·8118	0·0222	0·000364	0·000019	—	—
8	7·2578	0·0185	0·000242	—	—	—
9	7·6781	0·0157	—	—	—	—
10	8·0766	—	—	—	—	—

For completeness, the variances of the V_i have been included in this table in the column $p = 0$.

2·13. The efficiency of estimation of trend components using this approach is given in the following table:

m	1	2	3
1	100 %	73 %	49 %
2	100 %	66 %	39 %
3	100 %	63 %	34 %
4	100 %	61 %	31 %
5	100 %	59 %	28 %
6	100 %	58 %	27 %

It is again true that the efficiency decreases as m and p are increased. However, the loss in efficiency may be easily compensated by including one or two extra values of V_i, i.e. by increasing p.

2·14. The two processes of estimating the residual variance and the trend components may thus be carried out using differences of the quantities V_i. However, it must be remarked that this is not the only possible approximate basis for estimation. An alternative, which is likely to be somewhat better, would acknowledge the fact that the variance of V_m increases roughly linearly with $\sqrt{(m)}$ by extrapolating to zero using a suitably chosen scale instead of m.

One such scale appears to be provided by $\sqrt{(m+1)}-1$. If this is employed, with the method of divided differences, more efficient estimates can be obtained. The extra computation would, however, appear to be seldom worth while.

2·15. It might be asked what is the probable bias resulting from the failure to eliminate trend completely.

Suppose, for instance, that extrapolation is carried out using $V_m \dots V_{m+p}$, but that a trend component σ_t^2 is present in V_m. The error in the extrapolated estimate, V_0', is then

$$\sigma_t^2 + m\sigma_t^2 + \dots + \frac{m(m+1)\dots(m+p-1)}{p!}\sigma_t^2 = \frac{(m+1)(m+2)\dots(m+p)}{p!}\sigma_t^2.$$

However, the variance of σ_t^2 is

$$\frac{2\sigma^4}{N}\sum_{i=0}^{p}\sum_{j=0}^{p}(-1)^{i+j}\frac{\binom{p}{i}\binom{p}{j}\binom{4m+2i+2j}{2m+i+j}}{\binom{2m+2i}{m+i}\binom{2m+2j}{m+j}} = \frac{\lambda\sigma^4}{N}.$$

Thus the variance of the error in V_0' is

$$\left[\frac{(m+1)(m+2)\dots(m+p)}{p!}\right]^2\frac{\lambda\sigma^4}{N} = \frac{\lambda'\sigma^4}{N}, \quad \text{say.}$$

λ' is of order $\sqrt{(m)}/p$.

The following table gives a series of values for λ':

m	p				
	1	2	3	4	5
1	0·89	0·38	0·25	0·20	0·17
2	0·98	0·38	0·24	0·18	0·15
3	1·08	0·39	0·24	0·17	0·13
4	1·17	0·41	0·24	0·17	0·11
5	1·26	0·43	0·25	0·17	0·13
6	1·34	0·45	0·26	0·18	—
7	1·42	0·47	0·27	—	—
8	1·50	0·49	—	—	—
9	1·57	—	—	—	—

As a consequence of the fact that λ' decreases as p increases, we may say that the bias in the estimate of V_0 decreases with $\sqrt{(p)}$.

3. Examples

3·1. This theory may be illustrated using the two series given in the following table. The figures are in hundreds:

i	1	2	i	1	2	i	1	2	i	1	2
1	1000	1018	14	273	467	27	74	471	40	20	170
2	905	927	15	247	465	28	67	458	41	18	148
3	819	847	16	223	465	29	61	442	42	17	128
4	741	776	17	202	468	30	55	423	43	15	109
5	670	714	18	183	473	31	50	402	44	14	93
6	607	661	19	165	477	32	45	378	45	12	78
7	549	615	20	150	483	33	41	353	46	11	65
8	497	576	21	135	487	34	37	327	47	10	54
9	449	543	22	122	490	35	33	299	48	9	44
10	407	518	23	111	492	36	30	272	49	8	36
11	368	498	24	100	491	37	27	245	50	7	29
12	333	483	25	91	488	38	25	219	51	7	25
13	301	472	26	82	481	39	22	193	—	—	—

Both of these series consist of a trend plus a random element.

The trend in series 1 is given by

$$T_i = 10^5 \exp\left(-\frac{i-1}{10}\right) \quad (i = 1, \ldots, 51),$$

and that in series 2 by

$$T_i = 10^5\left[\exp\left(-\frac{i-1}{10}\right) + \frac{1}{\sqrt{(2\pi)}}\exp -\frac{(i-26)^2}{200}\right] \quad (i = 1, \ldots, 51).$$

In both series the random element consists of the error committed in reading the functions from three-place tables. Thus, in series 1,

$$u_i = 10^5\left[\exp\left(-\frac{i-1}{10}\right) \text{ to three decimal places } -\exp\left(-\frac{i-1}{10}\right)\right],$$

and, in series 2,

$$u_i = 10^5\left[\exp\left(-\frac{i-1}{10}\right) \text{ to three decimal places} - \exp\left(-\frac{i-1}{10}\right) + \frac{1}{\sqrt{(2\pi)}}\exp-\frac{(i-26)^2}{200} \text{ to three decimal places} - \frac{1}{\sqrt{(2\pi)}}\exp-\frac{(i-26)^2}{200}\right].$$

The value of σ^2 is consequently 816·6 for series 1 and 1633·3 for series 2.

Table 3·1*a*. *Estimates of the trend components in series* 1

V		Estimates				
		1st	2nd	3rd	4th	5th
1	2591687*	2582023·2	2573539·8	2565029·0	2556536·6	2548067·4
2	(±699)					
2	9663·8*	8483·4*	8510·8*	8492·4	8468·2	8442·6
	(±795)	(±42·1)	(±13·08)			
3	1180·4	−27·4	18·4	24·2*	25·6*	26·0*
	(±867)	(±33·1)	(±8·00)	(±3·12)	(±1·52)	(±0·84)
4	1207·8	−45·8	−5·8	−1·4	−0·4	−0·1
	(±924)	(±27·6)	(±5·46)	(±1·80)	(±0·75)	(±0·34)
5	1253·6	−40·0	−4·4	−1·0	−0·3	−0·3
	(±973)	(±23·9)	(±3·99	(±1·14)	(±0·42)	(±0·18)
6	1293·6	−35·6	−3·4	−0·7	0·0	—
	(±1015)	(±21·1)	(±3·06)	(±0·78)	(±0·26)	—
7	1329·2	−32·2	−2·7	−0·7	—	—
	(±1053)	(±19·0)	(±2·43)	(±0·56)	—	—
8	1361·4	−29·5	−2·0	—	—	—
	(±1087)	(±17·3)	(±1·98)	—	—	—
9	1390·9	−27·5	—	—	—	—
	(±1118)	(±16·0)	—	—	—	—
10	1418·4	—	—	—	—	—
	(±1146)	—	—	—	—	—

* Significant at upper 1 % level.

Table 3·1*b*. *Estimates of the trend components in series* 2

V		Estimates				
		1st	2nd	3rd	4th	5th
1	2173658·54*	2154432·32	2136689·82	2119064·59	2101548·55	2084135·64
2	19226·22*	17742·50*	17625·23	17516·04	17412·91	17311·91
3	1483·72	117·27	109·19*	103·13*	101·00*	100·23*
4	1366·45	8·08	6·06	2·13	0·77	0·36
5	1358·37	2·02	3·93	1·36	0·41	0·15
6	1356·35	−1·91	2·57	0·95	0·26	—
7	1358·26	−4·48	1·62	0·69	—	—
8	1362·74	−6·10	0·93	—	—	—
9	1368·84	−7·03	—	—	—	—
10	1375·87	—	—	—	—	—

3·2. Tables 3·1*a* and 3·1*b* give estimates of $V_1 - V_{10}$ for the two series together with the values of successive differences $\nabla^m V_i$ ($m = 1$ to 5).

The bracketed figures in table 3·1*a* give the standard errors of the estimates, calculated using $\sigma^2 = 816{\cdot}7$. Those for table 3·1*b* would be $\sqrt{(2)} = 1{\cdot}4$ times these values.

3·3. Examination of the estimates for series 1 shows a trend component of about 26 in V_3 and a negligible component in V_4. Similarly, for series 2, a trend component of about 100 exists in V_3 and a negligible component in V_4.

Estimators of V_0 may therefore be formed using values after V_3. Some possible combinations are given in the following table:

	Series 1	Series 2
$V_4 - V_9$	919·0	1549·1
$V_5 - V_{10}$	893·8	1522·6
$V_6 - V_{10}$	969·4	1484·9
$V_7 - V_{10}$	969·4	1421·9
$V_8 - V_{10}$	1053·4	1349·9

The coefficients of variation for those estimates vary between roughly

$$100\sqrt{(2{\cdot}8141)} = 26{\cdot}1\ \%$$

for the first pair and

$$100\sqrt{(4{\cdot}5141)} = 33{\cdot}1\ \%$$

for the last pair. None of these estimates differs from its theoretical value by more than its standard error.

3·4. To set more exact confidence limits, it is necessary to calculate the efficiency factors, E, of the estimators, using the values of λ given in the table of section 2·8. These are given by $E = 2/\lambda$.

Confidence limits may then be calculated in the same manner as for an ordinary estimate of variance based upon EN observations.

For instance, the efficiency value for the fourth estimate is $2/3{\cdot}92 = 0{\cdot}5102$, so that the effective number of observations is $41 \times 0{\cdot}5102 = 21$. The 95 % limits for σ^2 for series 1 are thus $969{\cdot}4/1{\cdot}69 = 573{\cdot}6$ and $969{\cdot}4 \times 2{\cdot}04 = 1977{\cdot}6$.

4. Correlation analysis for random series

4·1. For any two series it is possible to calculate not only the V's estimating the residual variances of the series, but also a set of C's estimating the residual covariances between the series. These C's may be calculated in an exactly similar manner to the V's except that each square has to be replaced by a product.

For example, if two series x_i and y_i ($i = 1, \ldots, N+10$) are used, it is necessary to replace $(\nabla^m x_i)^2$ by $(\nabla^m x_i)(\nabla^m y_i)$ in calculating C_m. Thus, for instance, we may set up a series of equations

$$C_{10} = \frac{(\nabla^{10}x_1)(\nabla^{10}y_1) + (\nabla^{10}x_2)(\nabla^{10}y_2) + \ldots + (\nabla^{10}x_N)(\nabla^{10}y_N)}{N},$$

$$C_9 = \frac{\frac{1}{2}(\nabla^9 x_1)(\nabla^9 y_1) + (\nabla^9 x_2)(\nabla^9 y_2) + \ldots + \frac{1}{2}(\nabla^9 x_{N+1})(\nabla^9 y_{N+1})}{N},$$

etc.

4·2. A curious analogue exists between the formulae for the covariances and those for the variances. The pth covariance may be written in the form

$$C_p = \mathrm{cov}_0(x,y) - \frac{2p}{p+1}\mathrm{cov}_1(x,y) + \frac{2p(p-1)}{(p+1)(p+2)}\mathrm{cov}_2(x,y) - \ldots,$$

where

$$\mathrm{cov}_0(x,y) = \mathrm{cov}(x_i y_i),$$

$$\mathrm{cov}_1(x,y) = \tfrac{1}{2}[\mathrm{cov}(x_i y_{i+1}) + \mathrm{cov}(x_{i+1}, y_i)],$$

$$\mathrm{cov}_2(x,y) = \tfrac{1}{2}[\mathrm{cov}(x_i y_{i+2}) + \mathrm{cov}(x_{i+2}, y_i)].$$

The formula for V_p is the particular case of this where $x_i = y_i$.

4·3. Now, if the correlation between x and y is ρ, and no serial or lagged correlation exists in the two series, it is not difficult to show that

$$\mathrm{var}[\mathrm{cov}_0(x,y)] = (1+\rho^2)\frac{\sigma_x^2\sigma_y^2}{N}$$

$$\mathrm{var}[\mathrm{cov}_i(x,y)] = \frac{1+\rho^2}{2}\frac{\sigma_x^2\sigma_y^2}{N} \quad (i = 1, 2, \ldots).$$

$$\mathrm{cov}[\mathrm{cov}_i(x,y), \mathrm{cov}_j(x,y)] = 0.$$

Since the factor $(1+\rho^2)$ occurs in all of these expressions, any process which minimizes the variance of a combination of the C_i's does so *irrespective of the value of* ρ. Consequently, the processes derived for the V_i (i.e. where $\rho = 1$) may also be applied to the C_i; the residual covariance between x and y may be estimated by extrapolation in the same manner as the residual variances were estimated.

4·4. For the same reason, correlation coefficients may be calculated using variances and covariances which have been estimated in similar fashions. The variances of these will be

$$\frac{(1-\rho^2)^2}{EN},$$

where E represents the efficiency of estimation.

The values of this factor are given in the following table, assuming that extrapolations involving V_{m+1} to V_{m+p} or C_{m+1} to C_{m+p} are used in estimating the correlation coefficient. It is derived by dividing the values of λ in the table of §2·8 into two (the minimum possible value for λ).

Efficiency of estimated correlation coefficient

m	1	2	3	4
0	0·667	0·781	0·826	0·851
1	0·514	0·649	0·692	0·727
2	0·433	0·567	0·631	0·671
3	0·381	0·509	0·575	0·617
4	0·344	0·466	0·532	0·575
5	0·316	0·433	0·496	0·539
6	0·294	0·406	0·467	0·510

4·5. It seems unlikely that the distribution of correlation coefficients calculated by the extrapolation method would differ very greatly from the distribution of ordinary correlation

coefficients, since, in the limit, the estimates are 100 % efficient. It would therefore seem to be reasonable to test any correlation calculated by the extrapolation method as if it were an ordinary correlation coefficient based upon En pairs of observations. This should give a good approximation to the true state of affairs.

5. Example

5·1. An example may be provided using series 1 and 2 of the last section. These series have a theoretical covariance of 816·7 and a theoretical correlation of 0·707.

Table 5·1a gives estimates of C_1 to C_{10} and their differences.

Table 5·1a. *Estimates of the covariances between series* 1 *and series* 2

p	C_p	Differences				
		1st	2nd	3rd	4th	5th
1	1561690·09	−1551285·16	1542029·34	−1532770·58	1523540·43	−1514343·00
2	10410·93	−9255·82	9258·76	−9230·15	9197·43	−9163·66
3	1155·11	2·94	28·61	32·72	33·77	−34·15
4	1158·05	31·55	−4·11	1·05	−0·38	0·19
5	1189·60	27·44	−3·06	0·67	−0·19	0·06
6	1217·04	24·38	−2·39	0·48	−0·13	—
7	1241·42	21·99	−1·91	0·35	—	—
8	1263·41	20·08	−1·56	—	—	—
9	1283·49	18·52	—	—	—	—
10	1302·01	—	—	—	—	—

This table shows similar characteristics to tables 3·1a and 3·1b, trend being largely eliminated in fourth and higher differences.

5·2. Estimates of covariances may be obtained in the same manner as estimates of variances. For example, using C_4–C_9, we get

$$C_0 = C_4 - 4\nabla C_4 + \frac{4\times 5}{2}\nabla^2 C_4 - \frac{4\times 5\times 6}{6}\nabla^3 C_4 + \dots$$

$$= 1158{\cdot}05 - 4\times 31{\cdot}55 - 10\times 4{\cdot}11 - 20\times 1{\cdot}04 - 35\times 0{\cdot}38 - 56\times 0{\cdot}19$$
$$= 945{\cdot}8.$$

The following table gives a group of estimates obtained in this fashion:

Estimate using	Estimated covariance
C_4–C_9	945·8
C_5–C_{10}	962·2
C_6–C_{10}	977·3
C_7–C_{10}	1004·6
C_8–C_{10}	1046·6

5·3. These estimates may be used to find the correlation between the two series. The following table gives a number of estimates of correlation coefficients together with the effective number of pairs of observations on which they are based.

Estimate using	Coefficients	No. of pairs
C_1*	0·658	27
C_2*	0·764	21
C_3*	0·873	18
C_4	0·901	16
C_5	0·912	14
C_6	0·919	13
C_7	0·924	12
C_8	0·928	11
C_9	0·930	11
C_{10}	0·932	10
C_4–C_9	0·793	29
C_5–C_{10}	0·811	27
C_6–C_{10}	0·815	23
C_7–C_{10}	0·856	21
C_8–C_{10}	0·878	18

* Biased by the existence of trend components.

It may be seen that, apart from the first three estimates, which are biased by the trend, the estimates obtained using serial estimates of covariance are better than those individually obtained.

5·4. Approximate limits may be set for any one coefficient by the use of the z transformation. For instance, the value of z corresponding to the coefficient 0·793 is 1·080. This has a standard error of roughly $1/\sqrt{(29)} = 0{\cdot}186$, and its 95 % confidence limits are consequently roughly 0·715 and 1·445. The corresponding 95 % confidence limits for ρ are 0·41 and 0·89.

It is perhaps worth noting that the actual sample correlation between the two series when their trend is removed is 0·781. The agreement with this calculated correlation coefficient is very good.

6. THE EFFECT OF SERIAL CORRELATION

6·1. The theory given above is based upon the assumption that the series used are serially independent. Where this assumption does not hold, the accuracy of the estimates of variance, covariance and correlation will differ from the values given above. For this reason, it is necessary to consider whether serial correlation exists in the series used and, if so, what modifications in the ordinary procedures need to be adopted.

In this and the next sections, attention will be largely concentrated upon the methods of answering three questions. These are:

(1) How may serial correlations in the series be detected?

(2) May any existing serial correlations be fitted into a scheme of the moving average or autoregressive type?

(3) What procedures exist for testing the correlation between series where both trend and serial correlation exist?

6·2. Before starting on a more serious consideration of these questions, one important point might be noted: If a large number of variate-differences are used in estimating the variances or covariances, these will give in general unbiased estimates of variances and covariances.

This will be apparent if it is remembered that

$$V_p = V_0\left[1-\frac{2p}{p+1}r_1+\frac{2p(p-1)}{(p+1)(p+2)}r_2-\dots\right],$$

and that the estimate of V_0 obtained from V_{m+1} to V_{m+p}, by either the most exact or by the extrapolation method, will be of the form

$$V'_0 = V_0[1-a_1r_1-a_2r_2-a_3r_3-\dots],$$

where $a_i \to 0$ as $p \to \infty$.

A more rigorous proof is outlined in Appendix 1. It indicates a desirable asymptotic independence of the serial correlation in the series.

6·3. Of course, in practice, we cannot let p become very large, but, even so, an appreciable reduction in bias may be achieved by the use of several of the V's (or C's).

For example, if $V_1, V_2, \dots, V_p$, are used, the most exact method gives an estimate

$$V'_0 = V_0\left[1-\frac{r_1}{p}-\frac{r_2}{p}-\dots-\frac{r_p}{p}\right],$$

while the extrapolation method gives an estimate

$$V'_0 = V_0\left[1-\frac{2}{p+1}r_1-\frac{4(p-1)}{(p+1)(p+2)}r_2-\frac{6(p-1)(p-2)}{(p+1)(p+2)(p+3)}r_3-\dots\right].$$

Both of these are liable to differ very little from V_0.

6·4. To illustrate the actual sizes of the bias, the following table gives the values that would be obtained for σ^2 by the extrapolation methods on different series for which $\sigma^2 = 1$:

Estimate using	Series			Moving average	Moving average
	Markoff	Auto-regressive	Auto-regressive		
	$\rho = 0{\cdot}5$	$a = 1{\cdot}1$, $b = 0{\cdot}5$	$a = 1{\cdot}1$, $b = 0{\cdot}8$	$\rho_1 = 0{\cdot}5$	$\rho_1 = -0{\cdot}64$, $\rho_2 = 0{\cdot}30$
V_1	0·500	0·267	0·389	0·500	1·640
V_5	0·365	0·067	0·039	0·167	2·352
V_{10}	0·349	0·058	0·031	0·091	2·648
V_1-V_{10}	0·796	0·841	1·179	0·909	1·035
V_2-V_{10}	0·668	0·594	1·071	0·818	1·096
V_5-V_{10}	0·477	0·175	0·201	0·545	1·445
V_8-V_{10}	0·386	0·078	0·047	0·273	2·040

These values are reasonably satisfactory for V_1–V_{10} and V_2–V_{10}, but are poor for V_8–V_{10}.

6·5. The first of the above queries may be reformulated to read: If $V_{m+1} \dots V_{m+p}$ may be assumed to be without any trend component and $\rho_{q+1}, \rho_{q+2} \dots$ may be assumed zero, how may we estimate $\rho_1, \rho_2 \dots \rho_q$?

Let us consider, first of all, the case $q = 1$.

One approach would be to express the means and variances of $V_{m+1} \dots V_{m+p}$ in terms of ρ_1 and σ_0, and thence to obtain maximum likelihood, or for rough uses least square, estimates of ρ_1 and σ. These would be quite complicated to obtain and would still provide only a first step towards simplifying the serial correlation picture; it would undoubtedly be necessary to reconsider these estimates as ρ_2 and higher serial correlations were taken into account. A simplification of the procedure, to allow more rapid approximate testing to be carried out, is therefore desirable.

One possible simplification is to ignore in the first instance the dependence of the variances of the V's on the ρ_i. It would then be possible to estimate $\sigma^2, \rho_1\sigma^2, \dots$ by least square procedure.

An equivalent method to this is to estimate $\rho_1 V_0$ from the differences

$$V_{m+1} - V_{m+2}, \quad V_{m+2} - V_{m+3}, \dots, V_{m+p-1} - V_{m+p},$$

and to use this to correct for the bias in the value estimated for V_0 on the assumption that $\rho_1 = 0$. This approach is the simplest mathematically and numerically.

6·6. We now consider a series of functions, $R_{11}, R_{12}, \dots$ defined by the formula

$$R_{1p} = \frac{(p+1)(p+2)}{1 \cdot 2} \nabla V_p.$$

It is not hard to show that

$$R_{1p} = V_0 \left[r_1 - \frac{4p}{p+3} r_2 + \frac{9p(p-1)}{(p+3)(p+4)} r_3 - \frac{16p(p-1)(p-2)}{(p+3)(p+4)(p+5)} r_4 + \dots \right],$$

so that if $\rho_2 = \rho_3 = \dots = 0$, we have

$$E(R_{1p}) = \rho_1 \sigma^2,$$

$$\frac{N}{\sigma^4} \operatorname{cov}(R_{1p} - R_{1q}) = \frac{1}{2} \frac{(2p+2q)!}{(p+q+2)!\,(p+q)!} (p \quad 2)!\, p!\, \frac{(q+2)!\, q!}{2q!} \frac{3pq + 2p + 2q + 1}{4pq + 2p + 2q + 1}.$$

6·7. Using this latter value, it is possible to set up equations similar to those of § 2·6. These may then be used to give the coefficients c_i and λ for estimating the first serial covariance and its variance. The following table gives the values of λ:

m	p			
	1	2	3	4
0	2·00	1·50	1·33	1·25
1	3·92	2·37	1·88	1·64
2	6·73	3·54	2·58	2·11
3	10·54	5·03	3·43	2·52
4	15·42	6·86	4·44	2·91
5	21·47	9·05	5·65	4·54

6·8. These values of λ might be compared with those obtained by extrapolation:

m	p		
	2	3	4
0	1·52	1·39	1·32
1	2·56	2·15	1·85
2	4·15	3·29	2·88
3	6·37	4·88	4·03
4	9·28	6·93	5·38
5	12·95	9·56	8·71

In this instance, direct extrapolation is not as efficient as previously, owing to the rapid increase in the variance of R_{1p} with p. This is of order $(p+1)^4$.

Better estimates may be obtained using the device suggested in §2·14 of using an alternative scale as a basis for extrapolation. In this instance, something like a quadratic scale is suggested, and the use of the scale $(m+3)^2$ appears to be fairly efficient.

6·9. It is fairly apparent that anything like a full analysis to estimate the higher serial correlation coefficients would involve a great deal of work which could not in general be justified. The prime need is for some fairly rapid method of gauging the likely serial correlations in any series using their variate differences. The above analysis gives an indication of how this may be approached.

In general, it is possible to define a series of functions

$$R_{mp} = \frac{(p+2m-1)(p+2m)}{(2m-1)2m} \nabla R_{m-1,p},$$

such that if the serial correlations higher than the mth are zero, the expectation of R_{mp} is $\rho_m \sigma^2$. These may be used for general investigational purposes.

The variances of the R_{mp} increase vary rapidly with p, so that the higher values are unreliable and, for extrapolation purposes, cubic, quartic and higher-order scales are required.

7. Examples

7·1. As a first illustration of these functions, series 1 and 2 may be used. Tables 7·1*a* and 7·1*b* give the estimates, R_{mp}, of serial correlation for these two series. Examination of these

Table 7·1*a*. *Estimates of serial covariance for series* 1

	0	1st	2nd	3rd	4th	5th
1	2591687	7746069·6	12825282·0	17776284·4	22551645·6	27102772·4
2	9663·8	50900·4	127936·0	236115·6	376650·0	545402·0
3	1180·4	− 274·0	1445·5	1755·6	4785·0	6921·2
4	1207·8	− 687·0	714·0	− 680·4	792·0	− 723·3
5	1253·6	− 840·0	940·8	− 1016·4	1149·7	− 5888·9
6	1293·6	− 996·8	1218·0	− 1429·1	3673·5	—
7	1329·2	− 1159·2	1542·8	− 2559·4	—	—
8	1361·4	− 1327·5	2035·0	—	—	—
9	1390·9	− 1512·5	—	—	—	—
10	1418·4	—	—	—	—	—

Table 7·1*b*. *Estimates of serial covariance for series* 2

	0	1st	2nd	3rd	4th	5th
1	2173658·54	6463296·96	10594736·62	14464143·22	17973894·37	21032116·20
2	19226·22	106455·00	263205·75	484447·60	765799·30	1098472·34
3	1483·72	1172·70	3680·25	7950·26	16840·88	28245·34
4	1366·45	121·20	367·64	− 623·28	545·49	233·08
5	1358·37	42·42	575·40	− 854·70	430·23	935·46
6	1356·35	− 53·48	808·50	− 1009·14	29·32	—
7	1358·26	− 161·28	1037·85	− 1018·16	—	—
8	1362·74	− 274·50	1233·65	—	—	—
9	1368·84	− 386·65	—	—	—	—
10	1375·87	—	—	—	—	—

tables is sufficient to show that, in both instances, trend apparently exists in the third variate-difference and that the first three serial correlation coefficients are likely to be negligible. At first sight, the largest would appear to be the first serial correlation coefficient of series 2, but an extrapolation using R_{15}–R_{19} gives an estimate of 117·32 for the serial covariance. This is well within the bounds of chance variation.

7·2. This analysis was carried out for four other theoretical series. These were:

Series 3. Eighty terms of a Markoff process with $\rho = 0{\cdot}5$. In fact, the first eighty terms of Kendall's (1949) series 11.

Series 4. Eighty terms of a moving average process for which $\rho_1 = 0{\cdot}4$. This was determined by $u_i = 2\epsilon_i + \epsilon_{i-1}$, where ϵ_i was the random element from Kendall's series 11.

Series 5. Eighty terms of the autoregressive process $u_i = 1{\cdot}1u_{i-1} - 0{\cdot}8u_{i-2} + \epsilon_i$. In fact, the first eighty terms of Kendall's (1945) series 3 multiplied by 0·05.

Series 6. Series 5 plus a trend component determined from

$$T_i = \exp -\frac{i}{50} + \sin 5i^0 + \frac{(i-30)^2}{1000}.$$

The estimates of serial correlation for these series are given in Tables 7·2*a*–7·2*d*.

Table 7·2*a*. *Estimates of serial covariance for series* 3 ($\sigma^2 = 1{\cdot}33$)

p \ *m*	0	1st	2nd	3rd	4th	5th
1	0·812338	0·404673	0·267895	0·055272	0·035729	0·149848
2	0·677447	0·243936	0·228415	0·027483	− 0·086874	0·075819
3	0·636791	0·152570	0·213692	0·081538	− 0·138569	0·019698
4	0·621534	0·091515	0·179718	0·152082	− 0·149928	0·109313
5	0·615433	0·053004	0·129024	0·215688	− 0·203984	0·290341
6	0·612909	0·031500	0·070200	0·288913	− 0·328416	—
7	0·611784	0·022140	0·004538	0·389964	—	—
8	0·611169	0·021645	− 0·070455	—	—	—
9	0·610688	0·028050	—	—	—	—
10	0·610178	—	—	—	—	—

Table 7·2*b*. *Estimates of serial covariance for series* 4 ($\sigma^2 = 5{\cdot}0$)

ρ \ m	0	1st	2nd	3rd	4th	5th
1	3·561117	2·503398	0·273930	−0·306278	−0·239615	0·615038
2	2·726651	2·339040	0·492700	−0·119911	−0·742828	0·328054
3	2·336811	2·141960	0·556938	0·342293	−0·966501	0·200368
4	2·122615	1·982835	0·414316	0·834330	−1·082098	0·452106
5	1·990426	1·894053	0·136206	1·293402	−1·305667	1·215830
6	1·900233	1·871352	−0·216540	1·762103	−1·826737	—
7	1·833399	1·900224	−0·617018	2·324176	—	—
8	1·780615	1·967535	−1·063975	—	—	—
9	1·736892	2·064260	—	—	—	—
10	1·699360	—	—	—	—	—

Table 7·2*c*. *Estimates of serial covariance for series* 5 ($\sigma^2 = 0{\cdot}0905$)

ρ \ m	0	1st	2nd	3rd	4th	5th
1	0·02762389	0·04650171	0·03204585	0·00373254	−0·02157966	−0·02510146
2	0·01212332	0·02727420	0·02937975	0·02051672	−0·00104210	−0·01603533
3	0·00757762	0·01552230	0·01838865	0·02116514	0·00989108	−0·00416622
4	0·00602539	0·01026840	0·00956984	0·01612968	0·01229467	0·00632286
5	0·00534083	0·00821772	0·00419328	0·01091376	0·00916798	0·00983670
6	0·00494951	0·00751884	0·00121680	0·00762269	0·00495225	—
7	0·00468098	0·00735660	−0·00051563	0·00609892	—	—
8	0·00447663	0·00741285	−0·00168850	—	—	—
9	0·00431190	0·00756635	—	—	—	—
10	0·00417433	—	—	—	—	—

Table 7·2*d*. *Estimates of serial covariance for series* 6 ($\sigma^2 = 0{\cdot}0905$)

ρ \ m	0	1st	2nd	3rd	4th	5th
1	0·02943038	0·05195976	0·04107950	0·01625029	−0·00570086	−0·00601408
2	0·01211046	0·02731206	0·02947215	0·02068429	−0·00078025	−0·01561504
3	0·00755845	0·01552320	0·01839128	0·02116978	0·00986637	−0·00416615
4	0·00600613	0·01026855	0·00957054	0·01614690	0·01226992	0·00656084
5	0·00532156	0·00821772	0·00418824	0·01094148	0·00902555	0·00874398
6	0·00493024	0·00751968	0·00120420	0·00770154	0·00527813	—
7	0·00466168	0·00735912	−0·00054615	0·00607750	—	—
8	0·00445726	0·00741870	−0·00171490	—	—	—
9	0·00429240	0·00757460	—	—	—	—
10	0·00415468	—	—	—	—	—

Examination of these tables shows that in no case could we be in doubt about the existence of serial correlation. There is very little difference between Tables 7·2*c* and 7·2*d*, and, from these tables alone, there might easily be some doubt about the existence of a trend component in series 6.

Table 7·2*c* indicates what is likely to be a common characteristic of oscillatory autoregressive schemes: the direct estimates of serial covariances are greater in many cases than

the corresponding estimates of variance. This is due to the fact that the latter will tend to be greatly reduced by the serial correlation, while the former will not be reduced to the same extent and may even be increased.

7·3. As practical illustrations of this approach, it is convenient to consider Kendall's (1943) ungraduated sheep series for 1867–1939 and Davis's (1941) figures of monthly freight-car loadings for the United States.

Estimates of serial covariances for the former series are given in Table 7·3*a*.

Table 7·3*a*. *Estimates of serial covariance for Kendall's sheep series*

ρ \ m	0	1st	2nd	3rd	4th	5th
1	3177·8585	6065·7162	5180·0270	2607·2578	875·0051	1306·9226
2	1155·9531	2957·7000	3317·7000	1926·6983	−194·2952	− 946·0034
3	663·0031	1630·6200	2285·5402	2047·5931	450·7071	− 424·8454
4	499·9411	977·6085	1432·3764	1818·1422	695·8102	−1471·9133
5	434·7672	670·7607	826·3290	1522·9500	1423·6794	−2034·3657
6	402·8305	532·9492	410·9790	1011·8856	2295·5504	—
7	383·7966	478·1520	181·0050	305·5624	—	—
8	370·5146	458·4060	122·2430	—	—	—
9	360·3278	447·2930	—	—	—	—
10	352·1952	—	—	—	—	—

There would seem to be little doubt about the existence of either a trend component or serial correlation in this series.

Trend apparently may be regarded as being confined to the first two variate differences, in which case an estimate of σ^2 may be derived from the third and higher variate differences. This estimate is more than 3700.

There is a marked similarity between the general characteristics of these estimates and those of series 5 and 6. The large values taken by R_{1p} and R_{2p} indicate that a second-order autoregressive scheme is likely to be needed to represent this series.

7·4. The analysis of Davis's freight-car loadings requires special adjustment because of the existence of seasonal variation. This could be easily carried out by an extension of the analysis of variance technique. It required the extraction of seasonal components from the formulae of § 2·3, using $m = 11$ instead of 10 as usual. Starting with 167 observations, this gave us a value for N, $167 - 11 - 12 = 144$.

Table 7·4*a*. *Estimates for serial covariance for freight-car loadings*

p \ m	0	1st	2nd	3rd	4th	5th
1	900·290	835·608	415·660	265·937	320·355	483·472
2	621·754	586·212	225·705	16·772	− 75·213	62·040
3	524·052	495·930	216·720	63·571	−117·513	68·763
4	474·459	434·010	190·232	123·396	−157·184	− 65·781
5	445·525	393·246	149·100	190·080	−124·655	−108·848
6	426·799	368·396	97·260	234·828	− 78·006	—
7	413·642	355·428	43·890	258·830	—	—
8	403·769	350·640	− 5·885	—	—	—
9	395·977	351·175	—	—	—	—
10	389·592	—	—	—	—	—

The estimates of serial covariance for this series, when seasonal variation has been eliminated, are given in Table 7·4*a*.

There is clearly a high trend component in V_1, but apparently not much beyond this. There is also clearly a high first serial correlation coefficient, a low second serial correlation coefficient and not much beyond this. These values suggest the fitting of either a Markoff process or a moving average scheme of length two. There is a fair degree of resemblance between this table and Table 7·2*a*.

8. Correlation analysis for non-random series

8·1. Before discussing correlation analysis for non-random series, it might be noted that the serial covariances of the variate differences may be directly calculated using the V_i. If D_{pq} is used to denote the qth serial covariance for the pth variate difference, then

$$D_{pq} = \frac{\Sigma_{(m-p-q)}(\nabla^p x_i)\,(\nabla^p x_{i+q})}{N\binom{2p}{p}},$$

and the following identities apply:

$$D_{p1} = V_p - \frac{2p+1}{p+1}V_{p+1},$$

$$D_{p2} = V_p - 4\frac{2p+1}{p+1}V_{p+1} + 2\frac{(2p+1)(2p+3)}{(p+1)(p+2)}V_{p+2},$$

$$D_{p3} = V_p - 9\frac{2p+1}{p+1}V_{p+1} + 12\frac{(2p+1)(2p+3)}{(p+1)(p+2)}V_{p+2} - \frac{4(2p+1)(2p+3)(2p+5)}{(p+1)(p+2)(p+3)}V_{p+3}.$$

These may be used to calculate the serial correlations of the variate differences up to any order.

8·2. A direct application of these formulae occurs when we consider the effect on the V_i of any linear transformation of the x_i. For instance, if the transformation

$$y_i = (x_i - \rho x_{i-1})/\sqrt{(1+\rho^2)}$$

is carried out, the revised V_i are given by

$$V'_i = V_i - \frac{2\rho}{1+\rho^2}D_i.$$

Similarly, a transformation of the type

$$y_i = (x_i - ax_{i-1} + bx_{i-2})/\sqrt{(1+a^2+b^2)}$$

gives values

$$V'_i = V_i - \frac{2a(1+b)}{1+a^2+b^2}D_{i1} + \frac{2b}{1+a^2+b^2}D_{i2}.$$

8·3. Correlation analysis may be carried out for non-random series in a similar fashion to that for random series. However, unless at least one of the series is random, the significance of the resulting coefficients is liable to be incorrectly estimated.

As an alternative approach to testing the correlation between non-random series, it is therefore often convenient to try to transform the series so that they consist of uncorrelated random elements. This is the approach which will be considered.

The major alternative to this approach is to adjust the degrees of freedom of the resulting estimates by a factor which would take into account the serial correlation in the series. This might be done using Bartlett's (1935) adjustment. (See, for example, Quenouille (1952).)

8·4. What the best method may be for determining a suitable transformation for serially correlated values when trend exists is not clear. However, one approach which appears to be reasonably satisfactory is to employ the transformation so that estimates of serial covariance of §6 disappear. This may be done using either the individual values or the extrapolates. The method will be clarified by examples.

9. EXAMPLES

9·1. It is not difficult to calculate the effect on Table 7·2*a* of adding D_{p1} to the estimates, V_p. This effect is given in Table 9·1*a*. The estimate of the first serial covariance from the first four values of Table 7·2*a* is 0·673841, while the corresponding estimate of the increment from Table 9·1*a* is 0·831860. Thus, 0·673841/0·831860 = 0·81 times Table 9·1*a* if subtracted from Table 7·2*a* gives a zero estimate for the first serial correlation coefficient.

Table 9·1*a*

p \ m	0	1st	2nd	3rd	4th	5th
1	−0·203832	0·540117	0·229965	0·151753	0·099939	0·054527
2	−0·383872	0·402138	0·121570	0·074023	0·055326	−0·019595
3	−0·450894	0·353510	0·082915	0·039598	0·056662	−0·086132
4	−0·486245	0·329820	0·066416	0·010752	0·106354	—
5	−0·508233	0·315588	0·062832	−0·034368	—	—
6	−0·523261	0·305116	0·075720	—	—	—
7	−0·534158	0·295020	—	—	—	—
8	−0·542353	—	—	—	—	—

This corresponds to a transformation $y_i = (x_i - \rho x_{i-1})/(1+\rho^2)$, where $2\rho/(1+\rho^2) = 0{\cdot}81$, i.e. $\rho = 0{\cdot}51$, and, if this is made in the above instance, all of the serial correlations may obviously be neglected.

9·2. The existence of trend in the above series would have necessitated the use of higher terms. For instance, if trend existed in V_1, the appropriate factor might have been estimated using the second to fifth terms. The estimate would have been 0·55982/0·685038 = 0·817, i.e. $\rho = 0{\cdot}52$.

9·3. For more complicated schemes, it is necessary to calculate the perturbations due to $D_{p1}D_{p2}$ and to consider these jointly. These may be used to set up simultaneous equations to find coefficients necessary to make the higher order serial covariances equal to zero.

For instance, for series 6, the perturbations due to D_{p1} and D_{p2} are as in Table 9·3*a*.

Since more than the values in table 9·3*a* would need to be subtracted from Table 7·2*d* to obtain zero estimates, it is apparent that a Markoff scheme would not suffice as a means of representing the series and that at least a second autogressive scheme must be used.

Table 9·3*a*

Perturbation due to D_{p1}

p \ m	0	1st	2nd	3rd	4th	5th
1	0·01126469	0·03525495	0·03410505	0·01768935	0·00511876	0·00283112
2	−0·00048696	0·01479192	0·02146980	0·01370809	0·00280239	−0·00792403
3	−0·00295228	0·00620400	0·01412618	0·01196438	0·00820514	−0·00277375
4	−0·00357268	0·00216795	0·00914102	0·00778722	0·00980538	—
5	−0·00371721	0·00020916	0·00654528	0·00362736	—	—
6	−0·00372717	−0·00088172	0·00555600	—	—	—
7	−0·00369568	−0·00162252	—	—	—	—
8	−0·00365061	—	—	—	—	—

Perturbation due to D_{p2}

p \ m	0	1st	2nd	3rd	4th	5th
1	−0·00544013	−0·00659004	0·01186480	0·02297309	0·02621533	0·02378685
2	−0·00324345	−0·01370892	−0·00454455	0·00258399	0·00675336	0·01197115
3	−0·00095863	−0·01189110	−0·00592883	−0·00161810	−0·00140879	0·00229327
4	0·00023048	−0·01019715	−0·00525462	−0·00090090	−0·00273183	—
5	0·00091029	−0·00907116	−0·00495432	0·00035806	—	—
6	0·00134225	−0·00824544	−0·00502470	—	—	—
7	0·00163673	−0·00757548	—	—	—	—
8	0·00184716	—	—	—	—	—

To estimate the appropriate proportions of Tables 9·3*a* and 9·3*b* to be added to Table 7·2*d*, extrapolates based upon the third to sixth variate differences might be used. These give equations

$$0{\cdot}06902037 + 0{\cdot}042869\,21A - 0{\cdot}023057\,61B = 0,$$

$$0{\cdot}07588594 + 0{\cdot}051247\,78A - 0{\cdot}010227\,22B = 0,$$

or

$$A = -1{\cdot}40,$$

$$B = 0{\cdot}38.$$

These values might be compared with the true values $A = -1{\cdot}39$, $B = 0{\cdot}56$ and with those obtained using extrapolates based upon the second to sixth variate differences, $A = -1{\cdot}39$, $B = 0{\cdot}11$.

9·4. The insensitivity of B in this instance is of no great concern so far as subsequent correlation analyses is concerned, since it is apparent that any inaccuracies in A and B will give rise to schemes for which the serial correlation coefficients are small. It is, however, preferable to choose values of A and B corresponding to schemes for which the coefficients a and b are real so that subsequent analysis may be carried out if required. This necessitates that both $1 + A + B$ and $8B(1 - B) - A$ should exceed zero. Thus, for instance, no scheme exists for which $A = -1{\cdot}40$, $B = 0{\cdot}38$, but, if $A = -1{\cdot}40$ and $B = 0{\cdot}44$, then both conditions are satisfied and $a = 1{\cdot}6$, $b = 1{\cdot}0$ gives a possible scheme.

The adaption of this latter scheme instead of the true scheme means that the quantities used in any correlation analysis are still serially correlated. The amount of this serial

correlation is, however, too small ($\rho_1 = 0{\cdot}06$, $\rho_2 = -0{\cdot}29$, $\rho_3 = -0{\cdot}27$, $\rho_4 = -0{\cdot}07$) to be of any importance.

The variate difference analysis may then be derived by the appropriate multiples of the above table from Table 7·2*d*. This gives the following results:

p \ m	0	1st	2nd	3rd	4th	5th
1	0·01126616	−0·00029679	−0·00144706	0·00159336	−0·00133238	0·00048857
2	0·01136509	0·00057145	−0·00258517	0·00262992	−0·00173212	0·00074591
3	0·01126984	0·00160552	−0·00399406	0·00370768	−0·00224069	0·00072614
4	0·01110929	0·00274667	−0·00553892	0·00484840	−0·00265962	—
5	0·01092618	0·00393359	−0·00715505	0·00597672	—	—
6	0·01073887	0·00512609	−0·00878507	—	—	—
7	0·01055579	0·00629744	—	—	—	—
8	0·01038086	—	—	—	—	—

This table indicates that very little serial correlation now exists in the series. The differences of the revised values of V_i may then be used to gain a better idea of the trend in the new series.

9·5. A similar analysis was carried out using Kendall's sheep series. This indicated clearly that a Markoff process could not be used to represent the series. The equations for determining A and B, derived using the third to sixth variate differences were

$$7437{\cdot}9107 + 4292{\cdot}2967A - 2408{\cdot}4016B = 0,$$

$$6891{\cdot}9200 + 5904{\cdot}1121A + 2639{\cdot}7281B = 0,$$

from which $A = -1{\cdot}42$, $B = 0{\cdot}56$.

The corresponding estimates using the second to sixth variate differences were $A = -1{\cdot}35$, $B = 0{\cdot}78$.

Both of these estimates are markedly greater than those for the artificial series, though the former estimate is very near the expected values for the artificial series. This serves to show that the differencing process $u_n - 1{\cdot}1u_{n-1} + 0{\cdot}8u_{n-2}$ will give rise to nearly random components which may then be used in a correlation analysis. The following table gives the values of V_i if $A = -1{\cdot}4$, $B = 0{\cdot}5$, i.e. if $a = 1{\cdot}21$, $b = 0{\cdot}76$. It may be seen that with these values there is very little evidence of serial correlation and trend does not seem to be of any importance after the first variate difference:

p \ m	0	1st	2nd	3rd	4th	5th
1	934·9359	72·5463	122·1597	211·8888	95·6965	457·7057
2	910·7538	−0·7502	− 29·1894	137·4583	−278·7898	658·7784
3	910·8788	10·9256	−102·8280	310·9275	−727·9566	−4415·6187
4	909·7863	40·3050	−232·3812	249·9381	1819·5162	—
5	907·0993	90·1909	−315·6938	−90·3925	—	—
6	902·8087	142·7166	−291·0414	—	—	—
7	897·7117	181·5221	—	—	—	—
8	892·6694	—	—	—	—	—

This latter statement can, of course, be tested using differences of the first column. In this instance there appear to be components of the order of 25·5 and 0·4 in V_1 and V_2.

9·6. A final analysis was carried out using Davis's freight-car loadings. Here again, the serial correlations could not be accounted for by a Markoff process, since a proportion of greater than one of D_{p1} would have been needed to account for the serial correlation. A second order process fitted using the third to sixth variate differences gave $A = -1{\cdot}03$, $B = 0{\cdot}38$, while one using the second to sixth variate differences gave $A = -1{\cdot}02$, $B = 0{\cdot}39$. The latter values correspond to $a = -0{\cdot}55$, $b = 0{\cdot}27$, which is not unlike a moving average process for which $\rho_1 = 0{\cdot}44$. The following table gives the values of V_i for this process.

	0	1st	2nd	3rd	4th	5th
1	851·388	40·775	91·896	147·772	116·311	232·453
2	837·796	−14·363	− 13·655	57·308	− 73·878	51·879
3	840·190	−8·900	− 44·356	103·275	−109·250	80·180
4	841·080	3·773	− 87·388	158·894	−155·507	—
5	840·828	22·499	−140·352	224·867	—	—
6	839·757	45·891	−201·680	—	—	—
7	838·118	72·781	—	—	—	—
8	836·096	—	—	—	—	—

There is apparently a low serial correlation in the revised series with a trend component of about 18 in the first variate difference.

10. Discussion

Since it is theoretically possible to reproduce any symmetrical derived second-moment statistics using the quantities V_i, results obtainable using any method of general trend elimination should be capable of reproduction using some modification of the variate-difference method. For example, the use of symmetrical moving average difference formulae is equivalent to using a smoothed second difference and, therefore, any results obtained by this method should be obtainable using some modification to the variate difference method.

Of course, as has been shown (Spencer-Smith, 1947; Quenouille, 1949), the conventional methods of eliminating trend are liable to lead to biased conclusions concerning the nature of the series, and, therefore, any alternative method which was unbiased or biased to a lesser degree would seem to have a claim for consideration.

The analysis described above is certainly more efficient for random series, while, for non-random series, it seems likely to be both less biased and more efficient than the usual method. It also has the added advantage of, in effect, reserving judgement on the extent and complexity of the trend until the serial correlation can be examined simultaneously. It cannot be denied that it requires fairly lengthy calculations, but these would not seem to be greatly more than the conventional analyses now employed and would seem more amenable to modern calculating equipment, whereby differencing and summing of squares can be carried out.

At this stage, many more practical problems will have to be tackled before the utility or otherwise of this method is established. It is interesting, however, to note that the old and, in many ways, discredited variate-difference method can by some slight modifications overcome many of the criticisms to which it is subject.

APPENDIX

It is necessary to consider the effect of extrapolation on

$$V_i = V_0\left[1-\frac{2i}{i+1}r_1+\frac{2i(i-1)}{(i+1)(i+2)}r_2-\dots\right]$$

when this is known for $i = m+1,\dots,m+p$.

It is not hard to show that if

$$f(i) = \frac{1}{a+i}-\frac{1}{a} \quad \text{for} \quad i = m+1,\dots,m+p,$$

then the value of $f(0)$ obtained by extrapolation is

$$-\frac{(m+1)(m+2)\dots(m+p)}{a(a+m+1)(a+m+2)\dots(a+m+p)}.$$

This follows, since $$\nabla^j f(m+1) = \frac{j!}{(a+m+1)(a+m+2)\dots(a+m+j+1)}$$

and $$f(0) = f(m+1)+(m+1)\nabla f(m+1)+\frac{(m+1)(m+2)}{2!}\nabla^2 f(m+1)+\dots.$$

As a corollary of this result, the extrapolate to zero may be found for any function which may be split into partial fractions. Thus the extrapolate to zero for the function

$$f(i) = \frac{i(i-1)\dots(i-q+1)}{(i+1)(i+2)\dots(i+q)} \quad \text{for} \quad i = m+1,\dots,m+p$$

equals

$$\sum_{k=1}^{q}\frac{(-1)^{k-1}(q+k-1)!}{(q-k)!\,k!\,(k-1)!}\frac{(m+k)!}{m!}\frac{(m+p)!}{(m+p+k)!} = \sum_{k=1}^{q}(-1)^{k-1}\binom{q-1}{k-1}\frac{(q+k-1)!\,(m+k)!\,(m+p)!}{(m+p+k)!\,m!\,k!\,(q-1)!}.$$

In this latter form it may be seen to be the $(q-1)$th difference of the function $\dfrac{(q+k-1)!\,(m+k)!\,(m+p)!}{(m+p+k)!\,m!\,k!\,(q-1)!}$.

If $q > m+p$, this function is a polynomial of order $q-p$ and, consequently (as might be expected), the extrapolate is zero.

If $q \leqslant m+p$, it may be noted that

$$\frac{(m+k)!}{k!\,(m+p+k)!} = \frac{1}{(p+k)!}-\frac{\binom{m}{1}p}{(p+k+1)!}+\frac{\binom{m}{2}p(p+1)}{(p+k+2)!}-\dots,$$

and that the $(q-1)$th difference of $\dfrac{(q+k-1)!}{(p+k+a)!}$ is $(-1)^{q-1}\dfrac{(a+p-1)!}{(a+p-q)!}\dfrac{q!}{(a+p+q)!}$. Using these two facts, the extrapolate may be put into the alternative form

$$(-1)^{q-1}\frac{q(m+p)!\,(p-1)!}{m!\,(p-q)!\,(p+q)!}\left[1-\frac{\binom{m}{1}p^2}{[(p+1)^2-q^2]}+\frac{\binom{m}{2}p^2(p+1)^2}{[(p+1)^2-q^2][(p+2)-q^2]}-\dots\right].$$

The proof may be completed by noting that this expression is of order $1/p$, when m is zero and p is large, q being regarded as small, and that even when q is of the order of p ($= Xp$, say) the expression is of the order $(1+X)^{-p}$ at most. For other values of m, a similar result may be obtained by considering the modification to the expression for $m = 0$.

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MOMENTS OF THE RANK CORRELATION COEFFICIENT τ IN THE GENERAL CASE*

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1. Introduction

The mean and variance of τ are known in the general case and for a number of special cases. Some of these results are given below in terms of p, the probability of concordance. Two observations (x_i, y_i) and (x_j, y_j) are said to be concordant if $(x_i - x_j)(y_i - y_j) > 0$; then it can be seen easily that p is related to τ by the formula

$$\tau = 2p - 1 = p - q,$$

where $q = 1 - p$.

(a) *General case*

If we write p' for the sample estimate of p (and similarly use a prime to distinguish sample statistics from their corresponding parental values), we have the result of Hoeffding (1947) for the general case:

$$E(p') = p, \tag{1}$$

$$\sigma^2(p') = \frac{2}{n(n-1)}\{pq + 2(n-2)(k-p^2)\}, \tag{2}$$

where k is the probability that out of three members drawn from the population, one is concordant with the other two.

(b) *Special case: independence*

For this case, Kendall (1938) has obtained

$$E(p') = \tfrac{1}{2}, \tag{3}$$

$$\sigma^2(p') = \frac{2n+5}{18n(n-1)}. \tag{4}$$

(c) *Special case: normal correlation*

For this case, the mean and variance, due to Greiner and Esscher respectively, are given by Kendall (1948):

$$E(p') = \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\rho, \tag{5}$$

$$\sigma^2(p') = \frac{1}{2n(n-1)}\left[1 - \left(\frac{2}{\pi}\sin^{-1}\rho\right)^2 + 2(n-2)\left\{\frac{1}{9} - \left(\frac{2}{\pi}\sin^{-1}\tfrac{1}{2}\rho\right)^2\right\}\right]. \tag{6}$$

Further, Hoeffding (1947) and Daniels & Kendall (1947) have proved that the distribution of t is asymptotically normal in the general case. Therefore, in the case of large samples, the above results are sufficient to fix the distribution completely. But it is notorious that the distribution of correlation coefficients in the non-null case tends to be extremely skew, particularly because of the limited range from -1 to $+1$; and that a very large sample is

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required before the skewness of the distribution is small enough to be neglected. The case of medium-size samples, which are likely to be most important in practice, therefore requires some consideration of the departure from normality of this sampling distribution. The third and fourth moments of this distribution are known only for the null case of independence. The cumulant-generating function of the distribution in this case was derived by Moran (1950) and Silverstone (1950), and from their result we obtain

$$\mu_3(p') = 0, \tag{7}$$

$$\mu_4(p') = \frac{100n^4 + 328n^3 - 127n^2 - 997n - 372}{2700n^3(n-1)^3}. \tag{8}$$

In this paper, formulae for the third and fourth moments are derived for the general case, and for a special case of normal correlation. But first we prove a conjecture of Daniels & Kendall (1947) about the upper bound of the variance.

2. Upper bound of the variance of t

Daniels & Kendall (1947) considered a population of ranked bivariate observations in which the sampling variance of t was maximized. They surmised that this occurred when the parent ranking was of the 'canonical' form, i.e. an arrangement where the X's are in the natural order, a certain number of the Y's at one end or the other of the sequence were in the correct natural order, and the others in the inverse order. A proof of this result is given below, in terms of p.

Let the parent population consist of N bivariate observations, such that the probability of concordance is p; this means that out of the $N(N-1)$ pairs (each pair counted twice) there are $P = pN(N-1)$ concordant pairs. Write P_i for the number of observations concordant with the member whose X-rank is i; then $P = \sum_i P_i$. The problem is then to find a subdivision of a given P into a set of P_i's such that the sampling variance of p' is a maximum. From (2) above, we see that in order to maximize the variance of p', for a given p, we have to maximize k.

Take two quantities P_i and P_j such that $P_i \geqslant P_j$. The contribution to k from these two is clearly $P_i(P_i-1)$ and $P_j(P_j-1)$. Now consider the effect of reducing P_j by one and increasing P_i by one. The contribution to k is then

$$P_i(P_i+1) + (P_j-1)(P_j-2), \quad \text{i.e.} \quad P_i(P_i-1) + P_j(P_j-1) + 2(P_i-P_j) + 2,$$

which is always greater than the original contribution; therefore the effect of such an alteration is always to increase the value of k. Therefore, k is maximized by increasing as many P_i as possible to their maximum value and reducing the other P_i as much as possible.

Now the maximum number of concordances with any single observation is clearly $(N-1)$, and for every observation with this number of concordances there is a contribution of 1 to every other P_i. Therefore the least possible value of P_i is equal to the number of observations with $(N-1)$ concordances. Further, in a ranking such as the following,

X-ranks	1	2	3	...	N
Y-ranks	Y_1	Y_2	Y_3	...	Y_N
Concordances	P_1	P_2	P_3	...	P_N

for any $P_i = (N-1)$, the rank Y_i must be the same as the X-rank i; all Y's to the left must be less than Y_i, and all Y's to the right must be greater than Y_i. Therefore all observations to

the left of (i, Y_i) must be concordant with the observations to the right of these, and so these concordances are not reduced to their least possible value. In order to do this, the maximum concordances must occur at one or the other end of the sequence. Therefore the ranking which maximizes k is one in which, for a given P, as many P_i as possible at one end are assigned their maximum value $(N-1)$ and the remainder are assigned their minimum value equal to the number of observations with $(N-1)$ concordances. It can be shown easily that this leads to a canonical ranking. This can be done in all cases, with the possible exception of one observation.

3. THIRD AND FOURTH MOMENTS IN THE GENERAL CASE

To find $$\mu_3(p') = E(p')^3 - 3E(p')^2 E(p') + 2E(p')$$

we first consider $(p')^3$. Arguing in the manner of Hoeffding (1947), $(p')^3$ is the probability of three pairs drawn from the sample, with replacement, being concordant. We may express this probability in terms of the following probabilities involving members of the sample drawn without replacement. Then $(p')^3$ is the sum of the following eight probabilities:

(i) The probability of drawing the same pair three times, $\binom{n}{2}^{-2}$, multiplied by the probability that it is concordant, p': $\binom{n}{2}^{-2} p'$.

(ii) The probability that two pairs are identical, but the third has one member in common with them, $3\binom{n}{2}^{-2} . 2(n-2)$, multiplied by the probability, termed k' by Hoeffding that among three members drawn from a sample, one is concordant with the other two:

$$6\binom{n}{2}^{-2}(n-2)\,k'.$$

(iii) The probability that two pairs are identical, while the third is distinct, $3\binom{n}{2}^{-2}\binom{n-2}{2}$, multiplied by the probability of drawing two concordant pairs, without replacement, i.e. $(p^2)'$, since the corresponding probability for the population is (p^2):

$$3\binom{n}{2}^{-2}\binom{n-2}{2}(p^2)'.$$

(iv) The probability that a pair has one member in common with the second pair, and the other member in common with the third, $3\binom{n}{2}^{-2} 2(n-2)(n-3)$, multiplied by the probability, l', that in a sample of four, two concordant members can be chosen in such a way that each of them is concordant with one of the remaining two:

$$6\binom{n}{2}^{-2}(n-2)(n-3)\,l'.$$

(v) The probability that two pairs have one member in common, while the third is completely distinct, $3\binom{n}{2}^{-2} 2(n-2)\binom{n-3}{2}$, multiplied by the probability, $(kp)'$, i.e. the

probability of getting a set of three members with the property mentioned in (ii) above, multiplied by the probability of a distinct concordant pair:

$$6\binom{n}{2}^{-2}\binom{n-3}{2}(n-2)(kp)'.$$

(vi) The probability of drawing three pairs, all with one member in common, and with the other members of each pair distinct, $2\binom{n}{2}^{-2}(n-2)(n-3)$, multiplied by the probability t' that among four members drawn from the sample without replacement, one is concordant with the other three:

$$2\binom{n}{2}^{-2}(n-2)(n-3)t'.$$

(vii) The probability of drawing three pairs, any two of which have only one member in common, i.e. if there are three members, A, B, C, the three pairs are AB, AC, BC, $2\binom{n}{2}^{-2}(n-2)$, multiplied by the probability that among three members drawn from the sample without replacement, each is concordant with the other two, u':

$$2\binom{n}{2}^{-2}(n-2)u'.$$

(viii) The probability of drawing three distinct pairs, $\binom{n}{2}^{-2}\binom{n-2}{2}\binom{n-4}{2}$, multiplied by the probability that they are all concordant, $(p^3)'$:

$$\binom{n}{2}^{-2}\binom{n-2}{2}\binom{n-4}{2}(p^3)'.$$

In a similar manner, $(p')^4$, the probability of drawing four concordant pairs from a sample, replacing each pair, is the sum of the following twenty-three probabilities.

(i) The probability of drawing the same pair four times, $\binom{n}{2}^{-3}$, multiplied by the probability that it is concordant, p':

$$\binom{n}{2}^{-3}p'.$$

(ii) The probability of drawing three identical pairs, and the fourth with one in common, $4\binom{n}{2}^{-3}2(n-2)$, multiplied by the probability, k', that among three members drawn from the sample, one is concordant with the other two:

$$8\binom{n}{2}^{-3}(n-2)k'.$$

(iii) The probability of drawing three identical pairs, and the fourth distinct, $4\binom{n}{2}^{-3}\binom{n-2}{2}$, multiplied by the probability that two pairs drawn without replacement are concordant, $(p^2)'$:

$$2\binom{n}{2}^{-3}(n-2)(n-3)(p^2)'.$$

(iv) The probability of drawing two sets of identical pairs, with one in common, $3\binom{n}{2}^{-3} 2(n-2)$, multiplied by the probability, k':

$$6\binom{n}{2}^{-3}(n-2)\,k'.$$

(v) The probability of drawing two identical pairs, and the third and fourth with the same member in common with the first two, and other members distinct, $6\binom{n}{2}^{-3} 2(n-2)(n-3)$, multiplied by the probability, t', that among four members drawn without replacement from a sample, one is concordant with the other three:

$$12\binom{n}{2}^{-3}(n-2)(n-3)\,t'.$$

(vi) The probability of drawing two identical pairs, the third with one member in common with the first two, and the other member in common with the fourth pair, $6\binom{n}{2}^{-3} 4(n-2)(n-3)$, multiplied by the probability, l', that among four members drawn from a sample, two concordant members can be chosen in such a way that each of them is concordant with one of the remaining two:

$$24\binom{n}{2}^{-3}(n-2)(n-3)\,l'.$$

(vii) The probability of drawing two identical pairs, with one member in common with the third pair, and the other member in common with the fourth pair, $6\binom{n}{2}^{-3} 2(n-2)(n-3)$, multiplied by the probability, l' (as in (vi)):

$$12\binom{n}{2}^{-3}(n-2)(n-3)\,l'.$$

(viii) The probability of drawing two identical pairs, with one member in common with third pair, and a fourth pair distinct, $12\binom{n}{2}^{-3} 2(n-2)\binom{n-3}{2}$, multiplied by the probability, $(kp)'$, as in (i) and (ii):

$$12\binom{n}{2}^{-3}(n-2)(n-3)(n-4)\,(kp)'.$$

(ix) The probability of drawing two identical pairs, with the same member in common with the third and fourth pairs, whose other members are common, $6\binom{n}{2}^{-3} 2(n-2)$, multiplied by the probability, u', that among three members drawn from the sample without replacement, each is concordant with the other two:

$$12\binom{n}{2}^{-3}(n-2)\,u'.$$

(x) The probability of drawing each of two distinct pairs twice, $3\binom{n}{2}^{-3}\binom{n-2}{2}$, multiplied by the probability, $(p^2)'$, that two pairs drawn without replacement are concordant:

$$\frac{3}{2}\binom{n}{2}^{-3}(n-2)(n-3)(p^2)'.$$

(xi) The probability of drawing two identical pairs, and two other pairs with one in common, $6\binom{n}{2}^{-3}\binom{n-2}{2}2(n-4)$, multiplied by the probability, $(kp)'$, as in (viii):

$$6\binom{n}{2}^{-3}(n-2)(n-3)(n-4)(kp)'.$$

(xii) The probability of drawing two identical pairs, and two distinct pairs,

$$6\binom{n}{2}^{-3}\binom{n-2}{2}\binom{n-4}{2},$$

multiplied by the probability, $(p^3)'$, of getting three concordant pairs without replacement:

$$\frac{3}{2}\binom{n}{2}^{-3}(n-2)(n-3)(n-4)(n-5)(p^3)'.$$

(xiii) The probability of drawing three pairs with one member in common and others distinct, and the fourth pair having the uncommon members of the others,

$$2\binom{n}{2}^{-3}4(n-2)(n-3).3,$$

multiplied by the probability, a', that among four members drawn from the sample without replacement, one is concordant with the other three, two of which are concordant with each other:

$$24\binom{n}{2}^{-3}(n-2)(n-3)a'.$$

(xiv) The probability of drawing two distinct pairs, the third pair having one member each from the first two pairs; and the fourth pair having the other members, $3\binom{n}{2}^{-3}4\binom{n-2}{2}$, multiplied by the probability, b', that among four members drawn from a sample without replacement, each is concordant with two other members:

$$6\binom{n}{2}^{-3}(n-2)(n-3)b'.$$

(xv) The probability of drawing three pairs with one in common, the fourth pair having one member in common with the uncommon members of the first three pairs, and another member distinct, $4\binom{n}{2}^{-3}2(n-2)(n-3)(n-4)$, multiplied by the probability, w', that among five members drawn without replacement, one is concordant with the others, one of which is concordant with the fifth:

$$24\binom{n}{2}^{-3}(n-2)(n-3)(n-4)w'.$$

(xvi) The probability of drawing a pair with one member in common with a second pair, and other member in common with a third pair, while the fourth pair has an uncommon member of the second or third, and a distinct member, $6\binom{n}{2}^{-3} 4(n-2)(n-3)(2n-4)$, multiplied by the probability, x', that among five members drawn without replacement from a sample, one is concordant with a second, which is concordant with a third, which is concordant with a fourth, which is concordant with the fifth:

$$24\binom{n}{2}^{-3}(n-2)(n-3)(n-4)x'.$$

(xvii) The probability of drawing one pair with one member in common with a second pair and other member in common with third pair, while a fourth pair is distinct,

$$6\binom{n}{2}^{-3} 4(n-2)(n-3)\binom{n-4}{2},$$

multiplied by the probability, $(lp)'$, as in (vi) and (i):

$$12\binom{n}{2}^{-3}(n-2)(n-3)(n-4)(n-5)(lp)'.$$

(xviii) The probability of drawing three pairs, any two of which have a member in common, while a fourth pair is distinct, $4\binom{n}{2}^{-3} 2(n-2)\binom{n-3}{2}$, multiplied by the probability, $(up)'$, as in (ix) and (i):

$$4\binom{n}{2}^{-3}(n-2)(n-3)(n-4)(up)'.$$

(xix) The probability of drawing three distinct pairs, one of which has a member in common with the fourth pair, $12\binom{n}{2}^{-3}\binom{n-2}{2}\binom{n-4}{2}(n-6)$, multiplied by the probability, $(kp^2)'$, as in (ii) and (iii):

$$3\binom{n}{2}^{-3}(n-2)(n-3)(n-4)(n-5)(n-6)(kp^2)'.$$

(xx) The probability of drawing four pairs with one member in common and other members distinct, $2\binom{n}{2}^{-3}(n-2)(n-3)(n-4)$, multiplied by the probability, y', that among five members drawn from a sample without replacement, one is concordant with other four:

$$2\binom{n}{2}^{-3}(n-2)(n-3)(n-4)y'.$$

(xxi) The probability of drawing three pairs with one member in common, and a fourth pair distinct, $4\binom{n}{2}^{-3} 2(n-2)(n-3)\binom{n-4}{2}$, multiplied by the probability, $(tp)'$, as in (v) and (i):

$$4\binom{n}{2}^{-3}(n-2)(n-3)(n-4)(n-5)(tp)'.$$

(xxii) The probability of drawing two pairs with one member in common, and two other pairs with a member in common, $3\binom{n}{2}^{-3} 2(n-2)\binom{n-3}{2} 2(n-5)$, multiplied by the probability, $(k^2)'$, as in (ii):

$$6\binom{n}{2}^{-3}(n-2)(n-3)(n-4)(n-5)(k^2)'.$$

(xxiii) The probability of drawing four distinct pairs, $\binom{n}{2}^{-3}\binom{n-2}{2}\binom{n-4}{2}\binom{n-6}{2}$, multiplied by the probability $(p^4)'$, of drawing four concordant pairs without replacement:

$$\frac{1}{128}\binom{n}{2}^{-3}(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)(p^4)'.$$

Collecting these terms together, we get the following results:

$$\mu_2'(p') = \binom{n}{2}^{-2}\{p+(n-2)(6k+2u)+(n-2)(n-3)(p^2+6l+2t) + (n-2)(n-3)(n-4)3kp+\tfrac{1}{4}(n-2)(n-3)(n-4)(n-5)p^3\}, \tag{9}$$

$$\mu_3(p') = \binom{n}{2}^{-2}\{(6l+2t-18kp+10p^3)n^2+(6k+2u-6p^2-30l-10t+72kp-34p^3)n + (p+9p^2+30p^3-12k-4u+36l+12t-72kp)\}, \tag{10}$$

$$\begin{aligned}\mu_4'(p') = \binom{n}{2}^{-3}\{&p+(n-2)(14k+12u)\\ &+(n-2)(n-3)(\tfrac{7}{2}p^2+12t+36l+24a+6b)\\ &+(n-2)(n-3)(n-4)(18kp+24w+24x+4up+2y)\\ &+(n-2)(n-3)(n-4)(n-5)(\tfrac{3}{2}p^3+12lp+4tp+6k^2)\\ &+(n-2)(n-3)(n-4)(n-5)(n-6)3kp^2\\ &+\tfrac{1}{8}(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)p^4\},\end{aligned} \tag{11}$$

$$\begin{aligned}\mu_4(p') = \binom{n}{2}^{-3}\{&6(k-p^2)^2n^4\\ &+(6kp+24w+24x+2y-6p^3-96lp-32tp-84k^2+270kp^2-108p^4)n^3\\ &+(\tfrac{3}{2}p^2+12t+36l+24a+6b-126kp-216w-216x\\ &\quad-24up-18y+75p^3+720lp+240tp+426k^2-1446kp^2+\tfrac{1011}{2}p^4)n^2\\ &+(14k+12u-\tfrac{31}{2}p^2-60t-180l-120a-30b\\ &\quad+444kp+624w+624x+96up+52y-213p^3\\ &\quad-1776lp-592tp-924k^2+2988kp^2-\tfrac{1857}{2}p^4)n\\ &+(p-28k-24u+21p^2+72t+216l+144a+36b-432kp-576w\\ &\quad-576x-96up-48y+180p^3+1440lp+480tp+720k^2\\ &\quad-2160kp^2+630p^4)\}.\end{aligned} \tag{12}$$

Considering only the dominant terms, we find

$$\sigma^2(p') \sim 4(k-p^2)n^{-1}, \tag{13}$$

$$\mu_3(p') \sim \text{const.}\, n^{-2}, \tag{14}$$

$$\mu_4(p') \sim 48(k-p^2)^2n^{-2}, \tag{15}$$

so that $\beta_1 \to 0$ and $\beta_2 \to 3$ as $n \to \infty$.

This agrees therefore with the general result for higher moments given by Hoeffding (1947) in his proof of the asymptotic normality of the distribution in the non-null case.

These formulae for the third and fourth moments involve ten parameters, namely, $p, k, l, t, u, a, b, w, x$ and y. In the general case when nothing is known about the parent population, we have no information about these parameters, except that provided by the sample itself. Then we can use sample estimates of these quantities, suitably corrected for bias where necessary. These computations may, however, be much too tedious for common applications. It is therefore interesting to study the values of these parameters in special cases.

4. SPECIAL CASE OF INDEPENDENCE

We first evaluate the ten parameters in the null case of independence, partly because it provides a check on our algebra, and partly because it is useful to have these values in extending to the non-null case.

First we collect some preliminary results. Let X_i $(i = 1, 2, \ldots)$ be a series of random variables, independently and identically distributed according to some common distribution function $F(x)$. Define $u_{ij} = X_i - X_j$. Then we need the values of quantities like

$$E(\operatorname{sgn} u_{12} \operatorname{sgn} u_{13}),$$

where

$$\begin{aligned} \operatorname{sgn} u_{ij} &= 1 \quad \text{if} \quad u_{ij} > 0, \\ &= 0 \quad \text{if} \quad u_{ij} = 0, \\ &= -1 \quad \text{if} \quad u_{ij} < 0. \end{aligned}$$

We consider also $v_{ij} = Y_i - Y_j$, where Y_i $(i = 1, 2, \ldots)$ is a series of independently and identically distributed random variables.

Obviously, $E(\operatorname{sgn} u_{ij}) = 0$. Given X_1, the probability that both X_2 and X_3 are less than X_1 is $\{F(x_1)\}^2$; and the probability that both X_2 and X_3 are greater than X_1 is $\{1 - F(x_1)\}^2$. Therefore

$$\Pr.\{(X_1 - X_2)(X_1 - X_3) > 0\} = \int_0^1 \{F(x)\}^2 dF(x) + \int_0^1 \{1 - F(x)\}^2 dF(x) = \tfrac{2}{3},$$

$$\Pr.\{(X_1 - X_2)(X_1 - X_3) < 0\} = \tfrac{1}{3},$$

so that

$$E(\operatorname{sgn} u_{12} \operatorname{sgn} u_{13}) = \tfrac{1}{3}.$$

We can express the ten parameters in terms of the u's and the v's. Under the null hypothesis, every u is independent of any v. Hence, we can evaluate all these parameters. As an example, we evaluate the quantity k by this method:

$$\begin{aligned} k &= E\{\tfrac{1}{2}(\operatorname{sgn} u_{12} \operatorname{sgn} v_{12} + 1)\, \tfrac{1}{2}(\operatorname{sgn} u_{13} \operatorname{sgn} v_{13} + 1)\} \\ &= \tfrac{1}{4} E(\operatorname{sgn} u_{12} \operatorname{sgn} u_{13})\, E(\operatorname{sgn} v_{12} \operatorname{sgn} v_{13}) \\ &\quad + \tfrac{1}{4}\{E(\operatorname{sgn} u_{12})\, E(\operatorname{sgn} v_{12}) + E(\operatorname{sgn} u_{13})\, E(\operatorname{sgn} v_{13})\} \\ &\quad + \tfrac{1}{4} = \tfrac{5}{18}. \end{aligned}$$

In this way, we obtain for the null case:

$$\left.\begin{array}{lll} p = \frac{1}{2}, & u = \frac{1}{6}, & x = \frac{76}{900}, \\ k = \frac{5}{18}, & a = \frac{7}{72}, & y = \frac{8}{75}. \\ l = \frac{11}{72}, & b = \frac{7}{72}, & \\ t = \frac{1}{6}, & w = \frac{163}{1800}, & \end{array}\right\} \tag{16}$$

However, this method of evaluating the parameters does not lend itself simply to generalization to the non-null case. We therefore consider an alternative procedure used by Hoeffding (1947) to evaluate k, and based on the consideration that in the null case, all permutations of one set of ranks relative to the other are equally probable. In the following, we write $P(1423)$ to indicate the probability that the ranks of the Y's occur in that order when the X's are arranged in order of magnitude.

(i) $p = \frac{1}{2}$,

(ii) $k = P(123) + \frac{1}{3}P\{(132) + (213)\} = \frac{5}{18}$,

(iii) $l = P(1234) + \frac{1}{2}P\{(1243) + (1324) + (2134)\}$
$+ \frac{1}{3}P(2143) + \frac{1}{6}P\{(1342) + (1423) + (2314) + (2413) + (3124)\}$
$= \frac{11}{72}$,

(iv) $t = P(1234) + \frac{1}{2}P\{(1243) + (1324) + (2134)\}$
$+ \frac{1}{4}P\{(1342) + (1423) + (1432) + (2314) + (3124) + (3214)\}$
$= \frac{1}{6}$,

(v) $u = P(123) = \frac{1}{6}$,

(vi) $a = P(1234) + \frac{1}{3}P\{(1243) + (1324) + (2134)\}$
$+ \frac{1}{12}P\{(1342) + (1423) + (2314) + (3124)\}$
$= \frac{7}{72}$,

(vii) $b = P(1234) + \frac{1}{3}P\{(1243) + (1324) + (2134) + (2143)\}$
$= \frac{7}{72}$,

(viii) $w = P(12345) + \frac{3}{5}P\{(12354) + (12435) + (13245) + (21345)\}$
$+ \frac{1}{3}P\{(12453) + (12534) + (13254) + (13425) + (14235)$
$+ (21354) + (21435) + (23145) + (31245)\}$
$+ \frac{1}{5}P\{(12543) + (14325) + (32145)\}$
$+ \frac{3}{20}P\{(13452) + (15234) + (23415) + (41235)\}$
$+ \frac{1}{3}P\{(13524) + (14253) + (21453) + (21534) + (23154) + (24135)$
$+ (31254) + (31425)\}$
$+ \frac{1}{12}P\{(13542) + (14352) + (15243) + (15324) + (24315)$
$+ (32415) + (41325) + (42135)\}$
$+ \frac{1}{10}P\{(21543) + (32154)\}$
$+ \frac{1}{15}P\{(14523) + (23514) + (24153) + (25134) + (31452)$
$+ (31524) + (34125) + (41253)\}$
$+ \frac{1}{30}P\{(14532) + (15342) + (15423) + (25143) + (25314) + (31542) + (32514)$
$+ (34215) + (41352) + (42153) + (42315) + (43125)\}$
$+ \frac{1}{60}P\{(24513) + (25413) + (34152) + (35124) + (35214) + (41523) + (41532)$
$+ (43152)\}$
$= \frac{163}{1800}$.

$$
\begin{aligned}
\text{(ix)}\quad x = {} & P(12345) + \tfrac{3}{5}P\{(12354)+(12435)+(13245)+(21345)\} \\
& + \tfrac{3}{10}P\{(12453)+(12534)+(13425)+(14235)+(23145)+(31245)\} \\
& + \tfrac{1}{10}P\{(12543)+(13452)+(14325)+(15234)+(21543)+(23415)+(32145) \\
& \quad +(32154)+(41235)\} \\
& + \tfrac{2}{5}P\{(13254)+(21354)+(21435)\} \\
& + \tfrac{1}{6}P\{(13524)+(14253)+(24135)+(31425)\} \\
& + \tfrac{1}{15}P\{(14523)+(23514)+(25134)+(31452)+(34125)+(41253)\} \\
& + \tfrac{7}{30}P\{(21453)+(21534)+(23154)+(31254)\} \\
& + \tfrac{7}{60}P\{(24153)+(31524)\} \\
& + \tfrac{1}{30}P\{(13542)+(14352)+(15243)+(15324)+(24315)+(24513)+(25143) \\
& \quad +(31542)+(32415)+(32514)+(34152)+(35124)+(41325)+(41523) \\
& \quad +(42135)+(42153)\} \\
& + \tfrac{1}{60}P\{(25314)+(35142)+(41352)+(42513)\} \\
= {} & \tfrac{76}{900},
\end{aligned}
$$

$$
\begin{aligned}
\text{(x)}\quad y = {} & P(12345) + \tfrac{3}{5}P\{(12354)+(12435)+(13245)+(21345)\} \\
& + \tfrac{2}{5}P\{(12453)+(12534)+(12543)+(13425)+(14235)+(14325)+(23145) \\
& \quad +(31245)+(32145)\} \\
& + \tfrac{1}{5}P\{(13254)+(13452)+(13524)+(13542)+(14253)+(14352) \\
& \quad +(14523)+(14532)+(15234)+(15243)+(15324) \\
& \quad +(15342)+(15423)+(15432)+(21354)+(21435)+(23415) \\
& \quad +(24135)+(24315)+(31425)+(32415)+(34125)+(34215) \\
& \quad +(41235)+(41325)+(42135)+(42315)+(43125)+(43215)\} \\
= {} & \tfrac{8}{75}.
\end{aligned}
$$

When we substitute these values for the ten parameters in (10) and (12), we get the same result as (7) and (8).

5. Special case of normal correlation

We now consider the values of the ten parameters in the case when the parent population is bivariate normal. For this case we already have, from (5)

$$p = \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\rho.$$

Further, by comparing (2) and (6) we have

$$k = \frac{1}{4}\left\{\frac{10}{9} + \frac{4}{\pi}\sin^{-1}\rho + \frac{4}{\pi^2}(\sin^{-1}\rho)^2 - \frac{4}{\pi^2}(\sin^{-1}\tfrac{1}{2}\rho)^2\right\}. \tag{17}$$

One other parameter can be evaluated exactly by the same methods used to deduce the above, i.e.

$$u = \frac{3}{8}\left\{\frac{4}{9} + \frac{4}{\pi}\sin^{-1}\rho - \frac{4}{\pi}\sin^{-1}\tfrac{1}{2}\rho + \frac{4}{\pi^2}(\sin^{-1}\rho)^2 - \frac{4}{\pi^2}(\sin^{-1}\tfrac{1}{2}\rho)^2\right\}. \tag{18}$$

The other parameters cannot be determined exactly; their evaluation involves the same type of difficulty as that pointed out by Kendall (1949) in evaluating the variance of Spearman's rank correlation coefficient r_s exactly under normal correlation. However, estimates of these parameters can be obtained fairly easily, because from the results of the previous section we see that only samples of five need be considered. I have shown in

another paper (Sundrum, 1953) how a systematic method of sampling may be used in such cases, and, in particular, used this method to obtain the frequency distribution of rankings of three, four and five from a bivariate normal population with correlation $\rho = 1/\sqrt{2}$. The theoretical values of three parameters (for the case $\rho = 1/\sqrt{2}$) and the sample estimates of the other seven parameters, obtained from the above sample results, are as follows:

$$\left.\begin{aligned} p &= 0{\cdot}7500, & a &= 0{\cdot}3470,\\ k &= 0{\cdot}5770, & b &= 0{\cdot}3485,\\ u &= 0{\cdot}4431, & w &= 0{\cdot}3451,\\ t &= 0{\cdot}4543, & x &= 0{\cdot}3363,\\ l &= 0{\cdot}4414, & y &= 0{\cdot}3642. \end{aligned}\right\} \tag{19}$$

Substituting these values in (10) and (12), we get

$$\mu_3 = \binom{n}{2}^{-2}\{-0{\cdot}013625n^2 + 0{\cdot}0007n - 0{\cdot}0427\}, \tag{20}$$

$$\mu_4 = \binom{n}{2}^{-3}\{0{\cdot}0012615n^4 - 0{\cdot}04316n^3 + 0{\cdot}59027n^2 - 1{\cdot}9285n + 1{\cdot}70314\}. \tag{21}$$

These formulae are not entirely satisfactory, and are given here only as an illustration of the method. Sampling experiments to improve these results and to obtain corresponding formulae for a range of values of ρ have been put in hand.

However, the argument of this paper shows that the third and fourth moments of t depend only on ten parameters, and that these parameters may be evaluated by finding the probability distribution of rankings of five from correlated populations.

In conclusion, the author would like to express his indebtedness to Prof. M. G. Kendall for suggesting the problem and for many valuable suggestions during the preparation of this paper.

[*Note added in proof.*] With regard to the argument of section 2 on the upper bound of the variance of t, it has been objected (I am indebted to Dr Hartley for pointing this out to me) that the transfer of units between the quantities designated P_i and P_j is not always possible; i.e. that it has not been proved that it is always possible to find a ranking in which the P_i's have the altered values. This objection may be met as follows:

The problem is essentially one of maximizing a certain function of the P_i's, subject to $P = \sum_i P_i$, and certain other constraints on the P_i's by virtue of their definition, and the object of showing that there is a ranking corresponding to the altered values of the P_i's is to show that the set of P_i's satisfies *all* the constraints at *every* stage of the maximization procedure. In section 2 only some of these constraints are taken into account during the maximization procedure. However, it is shown that the maximizing set of P_i's *does* satisfy all the constraints because it corresponds to an actual ranking, i.e. the canonical ranking.

Now, a quantity which is maximized subject to only a part of the set of all constraints must be greater or equal to the maximum subject to all constraints. If, however, the solution so obtained satisfies all constraints, then it must be equal to the maximum subject to all constraints.

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99·9 AND 0·1 % POINTS OF THE χ^2 DISTRIBUTION

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1. In answer to a special inquiry the lower 0·1 % points of the χ^2 distribution have been computed correct to 8 decimal places for degrees of freedom $n = 1(1)30(10)100, 120$, and also the upper 0·1 % points for $n = 40(10)100, 120$. The resulting values which it is thought may be of value to other statisticians are given in Tables 1 and 2 below.

Table 1. *Lower* 0·1 % *points of* χ^2

Degrees of freedom	Value of χ^2	Degrees of freedom	Value of χ^2	Degrees of freedom	Value of χ^2
1	0·0000 0157	16	3·9416 2784	40	17·9164 2654
2	0·0020 0100	17	4·4160 9272		
3	0·0242 9759	18	4·9048 4881	50	24·6739 0527
4	0·0908 0404	19	5·4068 1602		
5	0·2102 1260	20	5·9210 4075	60	31·7383 4159
6	0·3810 6676	21	6·4466 7656	70	39·0363 7739
7	0·5984 9375	22	6·9829 6844		
8	0·8571 0483	23	7·5292 3977	80	46·5198 7592
9	1·1519 4955	24	8·0848 8158		
10	1·4787 4346	25	8·6493 4363	90	54·1552 4356
11	1·8338 5266	26	9·2221 2682	100	61·9179 3921
12	2·2142 0932	27	9·8027 7692		
13	2·6172 1815	28	10·3908 7912	120	77·7551 4043
14	3·0406 7252	29	10.9860 5349		
15	3·4826 8447	30	11·5879 5105		

Table 2. *Upper* 0·1 % *points of* χ^2

Degrees of freedom	Value of χ^2	Degrees of freedom	Value of χ^2
40	73·4019 5752	80	124·8392 2402
50	86·6608 1519	90	137·2083 5413
60	99·6072 3307	100	149·4492 5278
70	112·3169 3185	120	173·6174 3646

Method of calculation

2. We write n for degrees of freedom, $m \equiv \frac{1}{2}n$ and $v \equiv \frac{1}{2}\chi^2$. If, then, $2v$ is the $100P$ % point of the χ^2 distribution for n degrees of freedom, and $r!$ is written for $\Gamma(r+1)$ whether r is integral or not,

$$\frac{1}{\Gamma(m)}\int_v^\infty e^{-t}t^{m-1}dt = P,$$

or, in other words,

$$f(v) = 0,$$

where $$f(v) = \left\{1 + v + \frac{v^2}{2!} + \ldots + \frac{v^{m-1}}{(m-1)!}\right\} - Pe^v \quad [n \text{ even}], \tag{1}$$

$$f(v) = \left\{\sqrt{\frac{2}{\pi}} e^v \int_{\sqrt{(2v)}}^{\infty} e^{-\frac{1}{2}t^2} dt + \frac{v^{\frac{1}{2}}}{(\frac{1}{2})!} + \frac{v^{\frac{3}{2}}}{(\frac{3}{2})!} + \ldots + \frac{v^{m-1}}{(m-1)!}\right\} - Pe^v \quad [n \text{ odd}]. \tag{2}$$

The exponential or quasi-exponential sum in curly brackets, i.e. $f(v) + Pe^v$, we write as S_{m-1} and the term $v^{m-1}/(m-1)!$ as T_{m-1}.

3. Each required value of $v(m, P)$ was obtained by computing $f(v)$ directly for an approximate value of v, $v = v_0$ say, and improving v_0 iteratively to $v_0 + \delta v$ by means of the formula

$$\delta v = \lambda + \alpha\lambda^2 + \beta\lambda^3, \tag{3}$$

where

$$\left.\begin{aligned} \lambda &= \frac{f(v)}{T_{m-1}}\bigg|_{v=v_0}, \\ \alpha &= \frac{1}{2}\left(\frac{T_{m-1} - T_{m-2}}{T_{m-1}}\right)_{v=v_0}, \\ \beta &= 2\alpha^2 - \frac{1}{6}\left(\frac{T_{m-1} - 2T_{m-2} + T_{m-3}}{T_{m-1}}\right)_{v=v_0}. \end{aligned}\right\} \tag{4}$$

This formula for δv follows from the Taylor expansion of v as a function of the small quantity $w \equiv e^{-v} f(v)$, thus:

$v = v_0$ when $w = e^{-v_0} f(v_0)$, $v = v_0 + \delta v$ when $w = 0$, hence

$$\delta v = -e^{-v_0} f(v_0) \left(\frac{dv}{dw}\right)_0 + \tfrac{1}{2} e^{-2v_0} f^2(v_0) \left(\frac{d^2v}{dw^2}\right)_0 - \tfrac{1}{6} e^{-3v_0} f^3(v_0) \left(\frac{d^3v}{dw^3}\right)_0 + \ldots.$$

But $w = e^{-v} S_{m-1} - P$, hence

$$\frac{dw}{dv} = e^{-v}(S_{m-2} - S_{m-1}) = -e^{-v} T_{m-1},$$

$$\frac{d^2w}{dv^2} = e^{-v}(T_{m-1} - T_{m-2}),$$

$$\frac{d^3w}{dv^3} = -e^{-v}(T_{m-1} - 2T_{m-2} + T_{m-3}),$$

whence the values of the inverse derivatives $d^r v/dw^r$ and the relations (3) and (4) readily follow.

Outline of procedure

4. For the first approximation v_0, use was made of the well-known formula

$$\sqrt[3]{(v/m)} \sim 1 - \frac{1}{9m} + x\sqrt{\frac{1}{9m}},$$

due to Wilson & Hilferty (1931), in which the $100P$ % value of v is related to the corresponding unit normal deviate x. The terms $T_r = v_0^r/r!$ were then computed in succession, starting from the greatest term T_q; the extra term

$$T_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi}} e^{v_0} \int_{\sqrt{(2v_0)}}^{\infty} e^{-\frac{1}{2}t^2} dt$$

required for odd n was obtained separately from Sheppard's (1939) 12-decimal table of $e^{\frac{1}{2}x^2} \int_x^{\infty} e^{-\frac{1}{2}t^2} dt$ and its derivatives.

The terms T_r were then added together to give the sum S_{m-1} and also the last few partial sums ..., S_{m-3}, S_{m-2}. Using the National Bureau of Standards (1951) 15- and 18-decimal tables of $e^{\pm x}$, e^{v_0} was obtained to the requisite number of places; the difference

$$f(v_0) = S_{m-1} - Pe^{v_0}$$

was then taken, and the correction δv computed from the relations (3) and (4).

For the next iteration, with $v = v_0 + \delta v$, no computation of terms T_r was necessary to obtain the new sum $S_{m-1}(v_0 + \delta v)$, this being calculated with ease from the first few terms of the Taylor series

$$S_{m-1}(v_0 + \delta v) = S_{m-1} + \delta v S_{m-2} + \tfrac{1}{2}\delta v^2 S_{m-3} + \tfrac{1}{6}\delta v^3 S_{m-4} + \ldots, \tag{5}$$

the S_r on the right-hand side being the values for $v = v_0$ already recorded. In the same way $T_{m-1}(v_0 + \delta v)$ was obtained from the Taylor series

$$T_{m-1} + \delta v T_{m-2} + \tfrac{1}{2}\delta v^2 T_{m-3} + \ldots.$$

After this, one further iteration at most was required, the sum $S_{m-1}(v_0 + \delta v + \delta v')$ being evaluated as above by using the accumulated correction $\delta v + \delta v'$ in the Taylor series (5).

Accuracy of method

5. In the above process, v is determined as the value for which the 'discrepancy' $f(v) = S_{m-1} - Pe^v$ comes out to zero. If in fact $S_{m-1} - Pe^v$ differs from 0 by a small amount ϵ, the error $\delta_1 v$ in v (so written because opposite in sign to the δv of paras. 3 and 4), is given by

$$\begin{aligned} \epsilon = \delta(S_{m-1} - Pe^v) &= (S_{m-2} - Pe^v)\,\delta_1 v \\ &\sim (S_{m-2} - S_{m-1})\,\delta_1 v = -T_{m-1}\delta_1 v, \end{aligned}$$

i.e.

$$\delta_1 v \sim -\epsilon / T_{m-1}.$$

To estimate ϵ, let us suppose that the pivotal term T_q is calculated to N significant figures, 1 unit in the last place being denoted by u, so that $10^{-(N-1)}T_q > u > 10^{-N}T_q$. Clearly T_q and the few terms T_r of like order (i.e. with the same number of figures before the decimal point) have errors of order u, the terms of next lower order (with one less figure before the decimal point) have errors of order $10^{-1}u$, and so on; the error in Pe^v is negligible. Hence ϵ is at most of order $10^{-(N-1)}T_q$.

When $P = 0{\cdot}001$, $T_q \equiv T_{m-1}$, hence $|\,\delta_1 v\,|$ is of order $10^{-(N-1)}$, and v is certainly correct to $N-2$ decimal places.

When $P = 0{\cdot}999$, $\qquad T_q/T_{m-1} \sim e^{\frac{1}{2}x^2} = e^{\frac{1}{2}(3{\cdot}090)^2} \sim 10^2$,

hence $|\,\delta_1 v\,|$ is of order $\qquad 10^{-(N-1)}T_q/T_{m-1}$, i.e. $10^{-(N-3)}$.

Thus v is certainly correct to $N-4$ decimal places.

Accordingly the terms T were calculated to 12 significant figures for $P = 0{\cdot}999$ and to 10 significant figures for $P = 0{\cdot}001$.

Checking of final results by differencing

6. It can readily be shown that $\chi^2_n(P)$, the $100P\,\%$ point of χ^2 for n degrees of freedom, is of the form

$$n + an^{\frac{1}{2}} + b + cn^{-\frac{1}{2}} + dn^{-1} + en^{-\frac{3}{2}} + \ldots, \tag{6}$$

where $a, b, c, \ldots$ are functions of P.

For this reason, direct differencing of the results $\chi^2_n(0{\cdot}999)$ against n for $n = 1(1)30$ and again for $n = 30(10)100$, and of $\chi^2_n(0{\cdot}001)$ for $n = 40(10)100$, was not practicable. An attempt to base the differencing check on the residuals

$$\{\chi^2_n(P) - \text{corresponding Wilson-Hilferty approximation}\}$$

also failed. In the end it was found possible to check all the results for $n \geqslant 14$ by a differencing method based on the accurate evaluation of series (6) as far as the term in $n^{-\frac{1}{2}}$, and to check the results for $n \leqslant 14$ by means of another series. Details of these series are given in paras. 7 and 8 respectively, and the steps in the checking procedure are outlined in para. 9.

7. *Series for* χ^2 *in terms of the corresponding normal deviate.* If x is the $100P$ % normal deviate, so that

$$\frac{1}{\sqrt{(2\pi)}}\int_x^\infty e^{-\frac{1}{2}t^2}\,dt = P = \frac{1}{\Gamma(\frac{1}{2}n)}\int_{\frac{1}{2}\chi^2}^\infty e^{-t}t^{\frac{1}{2}n-1}\,dt,$$

then the coefficients $a, b, c, \ldots$ in (6) can be expressed as polynomials in x. Developing the series for convenience in terms of $m \equiv \frac{1}{2}n$ instead of n, we may write

$$\chi^2 = 2m + a_0 x\, m^{\frac{1}{2}} + (b_0x^2 + b_1) + (c_0x^3 + c_1x)\, m^{-\frac{1}{2}} + (d_0x^4 + d_1x^2 + d_2)\, m^{-1} + \ldots, \qquad (7)$$

where $a_0, b_0, b_1, c_0, \ldots$ are constants. The values of these constants can be determined by equating moments about zero for the left- and right-hand sides of (7), bearing in mind that the rth moment of χ^2 is $2^r m(m+1)\ldots(m+r-1)$ and that of x, 0 when r is odd and $(r-1)(r-3)\ldots 3.1$ when r is even. For example, if we take (7) for convenience in the form

$$\chi^2 - 2m = a_0 x\, m^{\frac{1}{2}} + (b_0x^2 + b_1) + O(m^{-\frac{1}{2}})$$

and equate 1st, 2nd and 3rd moments each side, we get

$$2m - 2m = b_0 + b_1 + O(m^{-1}),$$

$$4m(m+1) - 4m.2m + 4m^2 = a_0^2 m + O(1),$$

$$8m(m+1)(m+2) - 6m.4m(m+1) + 12m^2.2m - 8m^3 = 3a_0^2(3b_0 + b_1)\,m + O(1),$$

whence $b_0 + b_1 = 0$, $a_0^2 = 4$, $3b_0 + b_1 = \frac{4}{3}$. Clearly the sign of a_0 is $+$, since $\chi^2 \gg 2m$ when $x \gg 0$; thus series (7) begins

$$\chi^2 = 2m + 2x\, m^{\frac{1}{2}} + \tfrac{2}{3}(x^2 - 1) + \ldots.$$

In this manner the expansion may be taken to any required number of terms; to order $m^{-\frac{1}{2}}$ it is

$$\chi^2 = 2m + 2x\, m^{\frac{1}{2}} + \tfrac{2}{3}(x^2 - 1) + \tfrac{1}{18}(x^3 - 7x)\, m^{-\frac{1}{2}} - \tfrac{1}{405}(3x^4 + 7x^2 - 16)\, m^{-1} + \tfrac{1}{19440}(9x^5 + 256x^3 - 433x)\, m^{-\frac{3}{2}} + \ldots. \qquad (8)$$

In particular, when $P = 0{\cdot}001$, $x = +3{\cdot}09023,\ 23062,$

and when $P = 0{\cdot}999$, $x = -3{\cdot}09023,\ 23062;$

hence, from (8),

$$\chi^2_{2m}(0{\cdot}001) = 2m + 6{\cdot}18046,\ 46124\, m^{\frac{1}{2}} + 5{\cdot}69969,\ 04708 + 0{\cdot}43770,\ 32003\, m^{-\frac{1}{2}} - 0{\cdot}80105,\ 59174\, m^{-1} + 0{\cdot}45024,\ 93635\, m^{-\frac{3}{2}} + \ldots \qquad (9)$$

$$= 2m + u_{\frac{1}{2}}m^{\frac{1}{2}} + u_0 + u_{-\frac{1}{2}}m^{-\frac{1}{2}} + u_{-1}m^{-1} + u_{-\frac{3}{2}}m^{-\frac{3}{2}} + \ldots, \text{ say;} \qquad (10)$$

and

$$\chi^2_{2m}(0{\cdot}999) = 2m - u_{\frac{1}{2}}m^{\frac{1}{2}} + u_0 - u_{-\frac{1}{2}}m^{-\frac{1}{2}} + u_{-1}m^{-1} - u_{-\frac{3}{2}}m^{-\frac{3}{2}} + \ldots. \qquad (11)$$

8. *Check series for $\chi^2_n(P)$ when n is small and P nearly* 1. It is clear from (1) and (2) that when P is near to unity, so that v/m is small,

$$(1-P)\,e^v = \frac{v^m}{m!} + \frac{v^{m+1}}{(m+1)!} + \dots$$

$$= \frac{v^m}{m!}\left(1 + \frac{v}{m+1} + \frac{v^2}{(m+1)(m+2)} + \dots\right);$$

hence

$$v^m = m!\,(1-P)\,e^v\left(1 + \frac{v}{m+1} + \dots\right)^{-1}$$

or

$$v = [m!\,(1-P)]^{1/m}\,e^{v/m}\left(1 + \frac{v}{m+1} + \dots\right)^{-1/m};$$

on expanding the exponential and binomial, multiplying together, and putting $2v = \chi^2$, this gives

$$\chi^2 = 2[m!(1-P)]^{1/m}\left\{1 + \frac{\chi^2}{n+2} + \frac{(\chi^2)^2}{2(n+2)(n+4)}\right.$$

$$+ \frac{(\chi^2)^3}{6(n+2)(n+4)(n+6)}\,\frac{n-2}{n+2}$$

$$+ \frac{(\chi^2)^4}{24(n+2)(n+4)(n+6)(n+8)}\,\frac{n^3-20n^2-124n-64}{(n+2)^2(n+4)}$$

$$\left. + \frac{(\chi^2)^5}{120(n+2)(n+4)(n+6)(n+8)(n+10)}\,\frac{(n-2)(n^3-136n^2-916n-704)}{(n+2)^3(n+4)} + \dots\right\}. \quad (12)$$

9. *Checking procedure.* The comparison between series (8) and the computed values of χ^2 furnished an adequate basis for a differencing check over the higher range of values of n, say $n \geqslant 30$, but not for n between say 15 and 30. On the other hand, the algebraic computation of further polynomial coefficients in (8) would have been laborious. The following method was therefore adopted:

Using equations (9), (10) and (11) and the set of paired results $\chi^2_n(0{\cdot}001)$, $\chi^2_n(0{\cdot}999)$ available for $n = 40(10)\,100$, the quantities

$$g_n \equiv \tfrac{1}{2}[\chi^2_n(0{\cdot}001) + \chi^2_n(0{\cdot}999)] - 2m - u_0 - u_{-1}m^{-1},$$

$$h_n \equiv \tfrac{1}{2}[\chi^2_n(0{\cdot}001) - \chi^2_n(0{\cdot}999)] - u_{\frac{1}{2}}m^{\frac{1}{2}} - u_{-\frac{1}{2}}m^{-\frac{1}{2}} - u_{-\frac{3}{2}}m^{-\frac{3}{2}},$$

were calculated, checked by differencing, and recomputed where necessary. Estimates $\hat{u}_{-2}, \hat{u}_{-3}, \hat{u}_{-4}$ of the coefficients of m^{-2}, m^{-3}, m^{-4} in (9) were then obtained by fitting the seven quantities g_n to the relationship

$$g_n = \hat{u}_{-2}m^{-2} + \hat{u}_{-3}m^{-3} + \hat{u}_{-4}m^{-4};$$

similarly, $\hat{u}_{-\frac{1}{2}}, \hat{u}_{-\frac{3}{2}}, \hat{u}_{-\frac{5}{2}}$ were obtained from the h_n.

The values of $\chi^2_n(0{\cdot}999)$ for $n = 14(1)\,30$ were now checked by differencing the residuals

$$\chi^2 - (2m - u_{\frac{1}{2}}m^{\frac{1}{2}} + u_0 - \dots - u_{-\frac{1}{2}}m^{-\frac{1}{2}} + \hat{u}_{-2}m^{-2} - \hat{u}_{-\frac{1}{2}}m^{-\frac{1}{2}} + \dots - \hat{u}_{-\frac{5}{2}}m^{-\frac{5}{2}}).$$

For $n = 1(1)\,14$ the values of $\chi^2_n(0{\cdot}999)$ were checked from equation (12) by evaluating for each given value of χ^2 the discrepancy between the two sides of the equation and differencing the quantities so obtained.

Standard approximations to χ^2 when n is large

10. It is of interest to compare the two most familiar approximations for χ^2 with the exact series obtained above.

I. $\sqrt{(2\chi^2)}$ distributed approximately normally with mean $\sqrt{(2n-1)}$ and variance 1 (Fisher).

This gives $$\chi^2 = n + x\sqrt{2n^{\frac{1}{2}}} + \tfrac{1}{2}(x^2-1) + O(n^{-\frac{1}{2}}).$$

II. $\sqrt[3]{(\chi^2/n)}$ distributed approximately normally with mean $1-2/(9n)$ and variance $2/(9n)$ (Wilson-Hilferty).

This gives $$\chi^2 = n + x\sqrt{2n^{\frac{1}{2}}} + \tfrac{2}{3}(x^2-1) + \frac{2\sqrt{2}}{27}(x^3-6x)\,n^{-\frac{1}{2}} + O(n^{-1}).$$

III. Exact series (equation (8)):

$$\chi^2 = n + x\sqrt{2n^{\frac{1}{2}}} + \tfrac{2}{3}(x^2-1) + \frac{\sqrt{2}}{18}(x^3-7x)\,n^{-\frac{1}{2}} + O(n^{-1}).$$

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TABLES OF SYMMETRIC FUNCTIONS. PART IV

By F. N. DAVID and M. G. KENDALL

1. In recent publications (David & Kendall, 1949, 1951) we have given, for functions up to and including weight 12, tables of the monomial symmetric functions (M) in terms of the one-part functions (S) and vice versa, tables of the unitary functions (U) in terms of M and vice versa, and tables of the homogeneous product sums (H) in terms of U. Pursuing our self-imposed task of compiling a definitive set of tables of symmetric functions we here present the MH-HM tables. In a further publication we shall complete the set by presenting the US-SU tables, the final tables HS-SH being equivalent to the US-SU tables with suitable changes of sign.

2. Many methods can be used for compiling and checking the present tables. One of the simplest is to start from the relationship

$$n!\,h_n = \sum \frac{n!}{1^{\pi_1}2^{\pi_2}\dots k^{\pi_k}\pi_1!\,\pi_2!\dots\pi_k!}\,s_1^{\pi_1}s_2^{\pi_2}\dots s_k^{\pi_k}.$$

The S-functions can be turned into monomial symmetric functions using Part I of our tables and the result follows. Conversely, we have

$$s_n = \sum(-1)^{(\pi_1+\pi_2+\dots+\pi_k+1)}\frac{(\pi_1+\pi_2+\dots+\pi_k-1)!\,n}{\pi_1!\,\pi_2!\dots\pi_k!}\,h_1^{\pi_1}h_2^{\pi_2}\dots h_k^{\pi_k}.$$

We take the monomial symmetric function and express it in terms of the S-functions from Part I tables. Substitution for s in terms of h gives the required result. The whole of the tables HM was checked by using MacMahon's D operator to build up each table successively. The converse tables MH were checked in the same way, first by using the operator and then by solving the simple equations of relationship which result. For the MH tables we have the additional check that the sum of the coefficients is zero.

3. In the introduction to Parts II and III we mentioned the uses of the U-functions in problems of distribution. The H-functions can be used in an analogous way and we quote therefore only one example. A distribution theorem due to MacMahon can be stated as follows:

'The number of ways of distributing objects of specification $(p_1p_2\dots p_r)$ into boxes of specification (1^m), no box being empty, is equal to the coefficient of

$$x^{p_1+p_2+\dots+p_r}(p_1p_2\dots p_r)$$

in the development of the function

$$(h_1x+h_2x^2+\dots)^m.'$$

When we come to use this theorem it is found that the easiest method of attack is the expression of the powers and products of the H-functions in terms of the M-functions. This is one of the simplest theorems in which the tables will be found of use.

4. In order to save space, when the tables have been set out use has been made of Meyer Hirsch's law of symmetry which states that the coefficient of the monomial symmetric function $(r_1r_2\dots r_l)$ in the expansion of $h_{l_1}h_{l_2}\dots h_{l_m}$ is equal to the coefficient of the monomial

symmetric function $(l_1 l_2 \dots l_m)$ in the expansion of $h_{r_1} h_{r_2} \dots h_{r_l}$. This enables each table to be written as a triangle and the only difficulty in putting the two parts together has been at the diagonal. In order to express the M-functions in terms of the H-functions we read horizontally until the diagonal in bold type is reached and then continue vertically downwards to the edge of the table. Thus for $w = 5$ we have

$$(31^2) = 3h_1^5 - 13h_2h_1^3 + 10h_2^2h_1 + 12h_3h_1^2 - 8h_3h_2 - 9h_4h_1 + 5h_5.$$

In order to express H-functions in terms of the M-functions we read vertically until the diagonal in bold type is reached and then continue horizontally to the right to the edge of the table. Thus, again for $w = 5$,

$$h_3h_1^2 = 20(1^5) + 13(21^3) + 8(2^21) + 7(31^2) + 4(32) + 3(41) + (5).$$

The two diagonal figures are separated by a bold horizontal rule.

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Table 4.2

$w=2$	h_1^2	h_2
(1^2)	$\frac{2}{1}$	1
(2)	−1	$\frac{1}{2}$

Table 4.3

$w=3$	h_1^3	h_2h_1	h_3
(1^3)	$\frac{6}{1}$	3	1
(21)	−2	$\frac{2}{5}$	1
(3)	1	−3	$\frac{1}{3}$

Table 4.4

$w=4$	h_1^4	$h_2h_1^2$	h_3^2	h_3h_1	h_4
(1^4)	$\frac{24}{1}$	12	6	4	1
(21^2)	−3	$\frac{7}{10}$	4	3	1
(2^2)	1	−4	$\frac{3}{3}$	2	1
(31)	2	−7	2	$\frac{2}{7}$	1
(4)	−1	4	−2	−4	$\frac{1}{4}$

Table 4.5

$w=5$	h_1^5	$h_2h_1^3$	$h_2^2h_1$	$h_3h_1^2$	h_3h_2	h_4h_1	h_5
(1^5)	$\frac{120}{1}$	60	30	20	10	5	1
(21^3)	−4	$\frac{33}{17}$	18	13	7	4	1
(2^21)	3	−14	$\frac{11}{14}$	8	5	3	1
(31^2)	3	−13	10	$\frac{7}{12}$	4	3	1
(32)	−2	10	−11	−8	$\frac{3}{11}$	2	1
(41)	−2	9	−7	−9	5	$\frac{2}{9}$	1
(5)	1	−5	5	5	−5	−5	$\frac{1}{5}$

Table 4.6

$w=6$	h_1^6	$h_2h_1^4$	$h_2^2h_1^2$	h_2^3	$h_3h_1^3$	$h_3h_2h_1$	$h_4h_1^2$	h_3^2	h_4h_2	h_5h_1	h_6
(1^6)	$\frac{720}{1}$	360	180	90	120	60	30	20	15	6	1
(21^4)	−5	$\frac{192}{26}$	102	54	72	38	21	14	11	5	1
(2^21^2)	6	−33	$\frac{58}{46}$	33	42	24	14	10	8	4	1
(2^3)	−1	6	−10	$\frac{21}{4}$	24	15	9	7	6	3	1
(31^3)	4	−21	26	−4	$\frac{34}{19}$	19	13	8	7	4	1
(321)	−6	34	−48	8	−30	$\frac{12}{61}$	8	6	5	3	1
(41^2)	−3	16	−20	4	−15	21	$\frac{7}{15}$	4	4	3	1
(3^2)	1	−6	9	−1	6	−15	−3	$\frac{4}{6}$	3	2	1
(42)	2	−12	19	−6	10	−20	−10	3	$\frac{3}{14}$	2	1
(51)	2	−11	14	−2	11	−17	−11	3	6	$\frac{2}{11}$	1
(6)	−1	6	−9	2	−6	12	6	−3	−6	−6	$\frac{1}{6}$

Table 4.7

$w=7$	h_1^7	$h_2h_1^5$	$h_2^2h_1^3$	$h_2^3h_1$	$h_3h_1^4$	$h_3h_2h_1^2$	$h_2^3h_1$	$h_3^2h_1$	$h_4h_1^3$	$h_4h_2h_1$	h_4h_3	$h_5h_1^2$	h_5h_2	h_6h_1	h_7
(1^7)	$\frac{5040}{1}$	2520	1260	630	840	420	210	140	210	105	35	42	21	7	1
(21^5)	−6	$\frac{1320}{37}$	690	360	480	250	130	90	135	70	25	31	16	6	1
(2^21^3)	10	−64	$\frac{378}{117}$	207	270	148	81	58	84	46	18	22	12	5	1
(2^31)	−4	27	−54	$\frac{120}{30}$	150	87	51	37	51	30	13	15	9	4	1
(31^4)	5	−31	53	−21	$\frac{208}{28}$	114	62	46	73	39	15	21	11	5	1
(321^2)	−12	78	−143	62	−70	$\frac{67}{195}$	39	30	43	25	11	14	8	4	1
(32^2)	3	−21	44	−26	17	−54	$\frac{25}{26}$	19	25	16	8	9	6	3	1
(3^21)	3	−20	37	−14	20	−60	13	$\frac{16}{25}$	20	13	7	8	5	3	1
(41^3)	−4	25	−43	18	−23	55	−14	−14	$\frac{34}{22}$	19	8	13	7	4	1
(421)	6	−40	76	−36	36	−101	30	29	−33	$\frac{12}{64}$	6	8	5	3	1
(43)	−2	14	−28	13	−14	45	−13	−19	11	−26	$\frac{4}{19}$	4	3	2	1
(51^2)	3	−19	33	−14	18	−43	12	10	−18	25	−7	$\frac{7}{18}$	4	3	1
(52)	−2	14	−29	17	−12	36	−17	−7	12	−24	7	−12	$\frac{3}{17}$	2	1
(61)	−2	13	−23	9	−13	33	−7	−10	13	−20	7	−13	7	$\frac{2}{13}$	1
(7)	1	−7	14	−7	7	−21	7	7	−7	14	−7	7	−7	−7	$\frac{1}{7}$

Table 4.8

$w=8$ (i)	h_1^8	$h_2h_1^6$	$h_2^2h_1^4$	$h_2^3h_1^2$	h_2^4	$h_3h_1^5$	$h_3h_2h_1^3$	$h_3h_2^2h_1$	$h_3^2h_1^2$	$h_3^2h_2$
(1^8)	**40320** / 1	20160	10080	5040	2520	6720	3360	1680	1120	560
(21^6)	−7	**10440** / 50	5400	2790	1440	3720	1920	990	680	350
(2^21^4)	15	−110	**2892** / 251	1548	828	2040	1092	584	412	220
(2^31^2)	−10	76	−183	**861** / 146	480	1110	618	345	248	139
(2^4)	1	−8	21	−20	**282** / 5	600	348	204	148	88
(31^5)	6	−43	94	−63	6	**1520** / 39	820	440	320	170
(321^3)	−20	148	−338	240	−24	−134	**463** / 489	260	194	108
(32^21)	12	−93	228	−182	20	81	−322	**154** / 258	116	69
(3^21^2)	6	−45	103	−70	6	44	−165	102	**92** / 66	54
(3^22)	−3	24	−61	50	−4	−22	94	−84	−33	**35** / 35
(41^4)	−5	36	−79	54	−6	−33	111	−66	−35	16
(421^2)	12	−90	209	−154	20	82	−298	192	99	−48
(42^2)	−3	24	−62	56	−12	−20	80	−64	−23	14
(431)	−6	46	−108	76	−8	−46	176	−107	−74	31
(4^2)	1	−8	20	−16	3	8	−32	20	14	−4
(51^3)	4	−29	64	−44	4	27	−91	58	27	−16
(521)	−6	46	−110	84	−8	−42	159	−119	−49	37
(53)	2	−16	40	−31	2	16	−67	54	27	−23
(61^2)	−3	22	−49	34	−4	−21	71	−43	−22	11
(62)	2	−16	41	−36	6	14	−56	48	15	−14
(71)	2	−15	34	−23	2	15	−53	31	19	−8
(8)	−1	8	−20	16	−2	−8	32	−24	−12	8

$w=8$ (ii)	$h_4h_1^4$	$h_4h_2h_1^2$	$h_4h_2^2$	$h_4h_3h_1$	h_4^2	$h_5h_1^3$	$h_5h_2h_1$	h_5h_3	$h_6h_1^2$	h_6h_2	h_7h_1	h_8
(1^8)	1680	840	420	280	70	336	168	56	56	28	8	1
(21^6)	1020	525	270	185	50	228	117	41	43	22	7	1
(2^21^4)	606	324	173	122	36	150	80	30	32	17	6	1
(2^31^2)	354	198	111	80	26	96	54	22	23	13	5	1
(2^4)	204	120	72	52	19	60	36	16	16	10	4	1
(31^5)	500	265	140	100	30	136	71	26	31	16	6	1
(321^3)	286	160	89	66	22	85	47	19	22	12	5	1
(32^21)	162	96	57	43	16	52	31	14	15	9	4	1
(3^21^2)	126	76	45	36	14	44	26	12	14	8	4	1
(3^22)	70	45	29	23	10	26	17	9	9	6	3	1
(41^4)	**209** / 31	115	63	47	16	73	39	15	21	11	5	1
(421^2)	−76	**68** / 211	40	31	12	43	25	11	14	8	4	1
(42^2)	20	−66	**26** / 36	20	9	25	16	8	9	6	3	1
(431)	40	−121	32	**17** / 98	8	20	13	7	8	5	3	1
(4^2)	−8	28	−12	−24	**5** / 10	8	6	4	4	3	2	1
(51^3)	−26	60	−14	−28	4	**34** / 26	19	8	13	7	4	1
(521)	39	−105	28	54	−8	−39	**12** / 76	6	8	5	3	1
(53)	−13	39	−8	−31	4	13	−31	**4** / 23	4	3	2	1
(61^2)	21	−50	14	23	−4	−21	29	−8	**7** / 21	4	3	1
(62)	−14	42	−20	−16	4	14	−28	8	−14	**3** / 20	2	1
(71)	−15	38	−8	−23	4	15	−23	8	−15	8	**2** / 15	1
(8)	8	−24	8	16	−4	−8	16	−8	8	−8	−8	**1** / 8

Table 4.9

$w=9$ (i)	h_1^9	$h_2h_1^7$	$h_2^2h_1^5$	$h_2^3h_1^3$	$h_2^4h_1$	$h_3h_1^6$	$h_3h_2h_1^4$	$h_3h_2^2h_1^2$	$h_3h_2^3$	$h_3^2h_1^3$	$h_3^2h_2h_1$	h_3^3	$h_4h_1^5$
(1^9)	**362880**	181440	90720	45360	22680	60480	30240	15120	7560	10080	5040	1680	15120
(21^7)	−8	**93240**	47880	24570	12600	32760	16800	8610	4410	5880	3010	1050	8820
(2^21^5)	21	−174	**25260**	13320	7020	17640	9300	4900	2580	3420	1800	660	5070
(2^31^3)	−20	170	−484	**7227**	3924	9450	5130	2787	1515	1980	1077	417	2880
(2^41)	5	−44	132	−154	**2202**	5040	2820	1584	894	1140	644	264	1620
(31^6)	7	−57	152	−146	36	**12840**	6840	3630	1920	2600	1370	510	4020
(321^4)	−30	250	−687	686	−177	−228	**3764**	2064	1128	1508	822	324	2250
(32^21^2)	30	−258	741	−789	222	231	−1110	**1173**	666	868	493	207	1250
(32^3)	−4	36	−111	134	−50	−30	153	−202	**396**	496	295	132	690
(3^21^3)	10	−84	231	−226	54	81	−374	392	−48	**664**	378	162	950
(3^221)	−12	105	−306	325	−82	−99	494	−597	78	−192	**227**	105	520
(3^3)	1	−9	27	−29	6	9	−48	63	−6	21	−45	**55**	210
(41^5)	−6	49	−131	127	−33	−45	195	−195	27	−68	78	−6	**1545**
(421^3)	20	−168	466	−474	132	154	−699	740	−112	249	−304	24	−143
(42^21)	−12	105	−309	344	−110	−93	451	−536	98	−154	222	−18	87
(431^3)	−12	102	−284	283	−74	−100	463	−477	66	−189	216	−18	91
(432)	6	−54	164	−188	60	50	−256	324	−60	96	−157	15	−44
(4^21)	3	−26	74	−76	22	26	−122	124	−19	52	−51	3	−26
(51^4)	5	−41	110	−107	27	38	−165	169	−23	56	−70	6	−36
(521^2)	−12	102	−287	297	−82	−94	434	−479	74	−151	204	−18	88
(52^2)	3	−27	83	−100	38	23	−115	148	−38	35	−51	3	−23
(531)	6	−52	148	−150	36	52	−250	275	−34	104	−142	15	−46
(54)	−2	18	−54	60	−19	−18	90	−104	19	−38	47	−3	18
(61^3)	−4	33	−89	87	−22	−31	135	−139	18	−46	60	−6	30
(621)	6	−52	150	−160	44	48	−229	268	−38	79	−127	15	−45
(63)	−2	18	−54	59	−15	−18	93	−117	15	−39	72	−12	15
(71^2)	3	−25	68	−67	18	24	−105	106	−16	37	−42	3	−24
(72)	−2	18	−55	65	−23	−16	80	−103	23	−25	41	−3	16
(81)	−2	17	−47	46	−11	−17	77	−78	9	−30	35	−3	17
(9)	1	−9	27	−30	9	9	−45	54	−9	18	−27	3	−9

$w=9$ (ii)	$h_4h_1^5$	$h_4h_2^2h_1$	$h_4h_2h_1^3$	$h_4h_3h_2$	$h_4^2h_1$	$h_5h_1^4$	$h_5h_2h_1^2$	$h_5h_2^2$	$h_5h_3h_1$	h_5h_4	$h_6h_1^3$	$h_6h_2h_1$	h_6h_3	$h_7h_1^2$	h_7h_2	h_8h_1	h_9
(1^9)	7560	3780	2520	1260	630	3024	1512	756	504	126	504	252	84	72	36	9	1
(21^7)	4515	2310	1575	805	420	1932	987	504	343	91	357	182	63	57	29	8	1
(2^21^5)	2670	1405	980	515	280	1206	634	333	232	66	246	129	47	44	23	7	1
(2^31^3)	1566	852	606	330	186	738	402	219	156	48	165	90	35	33	18	6	1
(2^41)	912	516	372	212	123	444	252	144	104	35	108	62	26	24	14	5	1
(31^6)	2115	1110	785	410	230	1044	543	282	197	56	229	118	42	43	22	7	1
(321^4)	1229	669	486	263	154	626	339	183	132	41	151	81	31	32	17	6	1
(32^21^2)	710	403	298	169	102	370	210	119	88	30	97	55	23	23	13	5	1
(32^3)	408	243	181	109	67	216	129	78	58	22	61	37	17	16	10	4	1
(3^21^3)	548	313	240	135	86	302	172	97	74	26	86	48	20	22	12	5	1
(3^221)	313	188	145	87	56	174	105	63	49	19	53	32	15	15	9	4	1
(3^3)	135	87	69	45	30	78	51	33	27	12	27	18	10	9	6	3	1
(41^5)	840	455	335	180	110	501	266	141	101	31	136	71	26	31	16	6	1
(421^3)	**479**	272	206	116	74	287	161	90	67	23	85	47	19	22	12	5	1
(42^21)	**520**	**163**	125	75	49	163	97	58	44	17	52	31	14	15	9	4	1
(431^3)	−344	**270**	**101**	60	42	127	77	46	37	15	44	26	12	14	8	4	1
(432)	−344	217	**277**	**39**	27	71	46	30	24	11	26	17	9	9	6	3	1
(4^21)	184	−154	−131	**112**	**21**	48	32	21	18	9	20	13	7	8	5	3	1
(51^4)	104	−69	−91	38	**39**	**209**	115	63	47	16	73	39	15	21	11	5	1
(521^2)	119	−71	−71	36	18	**35**	**68**	40	31	12	43	25	11	14	8	4	1
(52^2)	−317	209	200	−113	−55	−84	**228**	**26**	20	9	25	16	8	9	6	3	1
(531)	94	−80	−54	42	19	21	−68	**38**	**17**	8	20	13	7	8	5	3	1
(54)	175	−110	−143	74	42	42	−123	28	**97**	**5**	8	6	4	4	3	2	1
(61^3)	−76	57	67	−38	−29	−14	47	−19	−38	**29**	**34**	10	8	13	7	4	1
(621)	−99	61	56	−31	−13	−30	69	−16	−32	9	**30**	**12**	6	8	5	3	1
(63)	166	−120	−95	68	22	45	−121	32	62	−18	−45	**88**	**4**	4	3	2	1
(71^2)	−60	45	45	−36	−9	−15	45	−9	−36	9	15	−36	**27**	**7**	4	3	1
(72)	81	−49	−50	25	13	24	−57	16	26	−9	−24	33	−9	**24**	**3**	2	1
(81)	−64	55	34	−32	−9	−16	48	−23	−18	9	16	−32	9	−16	**23**	**2**	1
(9)	−60	35	43	−18	−13	−17	43	−9	−26	9	17	−26	9	−17	9	**17**	**1**
	36	−27	−27	18	9	9	−27	9	18	−9	−9	18	−9	9	−9	−9	**9**

Table 4.10

$w = 10$ (i)	h_1^{10}	$h_2h_1^8$	$h_3^2h_1^4$	$h_2^3h_1^4$	$h_4^2h_1^2$	$h_3h_1^7$	$h_3h_2h_1^3$	$h_2^4h_1^2$
(1^{10})	3628800 1	1814400	907200	453600	226800	113400	604800	302400	151200
(21^8)	−9	927360 82	473760	241920	123480	63000	322560	164640	84000
(2^21^6)	28	−259	247320 834	129060	67320	35100	171360	89400	46620
(2^31^4)	−35	330	−1090	68868 1476	36756	19620	90720	48420	25848
(2^41^2)	15	−145	496	−708	20100 371	11010	47880	26160	14316
(2^5)	−1	10	−36	56	−35	6210 6	25200	14100	7920
(31^7)	8	−73	230	−290	124	−8	122640 67	64680	34020
(321^5)	−42	390	−1256	1628	−720	48	−358	34960 1965	18850
(32^21^3)	60	−570	1892	−2555	1194	−84	518	−2946	10434 4655
(32^21)	−20	196	−680	982	−512	40	−172	1023	−1746
(3^21^4)	15	−140	451	−578	246	−15	134	−741	1102
(3^221^2)	−30	288	−963	1296	−582	30	−273	1584	−2531
(3^23^2)	6	−60	213	−315	166	−10	54	−333	596
(3^21)	4	−39	132	−177	74	−4	39	−234	383
(41^6)	−7	64	−202	256	−112	8	−59	313	−449
(421^4)	30	−280	907	−1188	543	−42	258	−1412	2106
(42^21^2)	−30	288	−969	1338	−660	60	−261	1491	−2368
(42^3)	4	−40	143	−218	130	−20	34	−204	352
(431^3)	−20	188	−610	790	−348	24	−182	1006	−1485
(4321)	24	−234	798	−1106	528	−40	222	−1306	2124
(43^2)	−3	30	−105	148	−68	4	−30	186	−320
(4^21^2)	6	−57	187	−246	114	−10	56	−310	452
(4^22)	−3	30	−106	156	−86	12	−28	168	−280
(51^5)	6	−55	174	−221	96	−6	51	−271	393
(521^3)	−20	188	−614	812	−372	24	−174	961	−1463
(52^21)	12	−117	402	−572	292	−20	105	−613	1020
(531^2)	12	−114	374	−489	214	−12	112	−629	950
(532)	−6	60	−212	310	−160	8	−56	342	−600
(541)	−6	58	−194	260	−120	8	−58	330	−500
(5^2)	1	−10	35	−50	25	−1	10	−60	100
(61^4)	−5	46	−146	186	−81	6	−43	229	−333
(621^2)	12	−114	377	−506	236	−20	106	−594	920
(62^2)	−3	30	−107	162	−94	12	−26	156	−272
(631)	−6	58	−194	258	−112	8	−58	336	−525
(64)	2	−20	70	−100	51	−6	20	−120	196
(71^3)	4	−37	118	−151	66	−4	35	−187	273
(721)	−6	58	−196	270	−128	8	−54	311	−503
(73)	2	−20	70	−99	46	−2	20	−123	212
(81^2)	−3	28	−90	116	−52	4	−27	145	−210
(82)	2	−20	71	−106	59	−6	18	−108	188
(91)	2	−19	62	−80	34	−2	19	−105	154
(10)	−1	10	−35	50	−25	2	−10	60	−100

Table 4.10 (*continued*)

$w=10$ (ii)	$h_6h_2^2h_1$	$h_5h_1^5$	$h_4h_3h_2h_1$	$h_4h_2^3$	$h_3^2h_1$	$h_3h_1^4$	$h_3h_2h_1^3$	$h_3h_2^2h_1^2$	$h_3h_2^3$	$h_2^5h_1$
(1^{10})	75600	100800	50400	25200	16800	151200	75600	37800	18900	25200
(21^8)	42840	57120	29120	14840	10080	85680	43680	22260	11340	15120
(2^21^6)	24300	32280	16820	8760	6060	48060	25050	13050	6795	9030
(2^31^4)	13800	18180	9708	5184	3648	26730	14274	7623	4071	5364
(2^41^2)	7848	10200	5596	3076	2196	14760	8088	4440	2442	3168
(2^5)	4470	5700	3220	1830	1320	8100	4560	2580	1470	1860
(31^7)	17850	24080	12600	6580	4620	36960	19320	10080	5250	7070
(321^5)	10140	13560	7280	3900	2790	20340	10930	5860	3135	4195
(32^21^3)	5760	7600	4200	2318	1686	11130	6154	3399	1875	2472
(32^21)	3288	4240	2418	1382	1017	6060	3450	1968	1125	1447
(3^21^4)	774	5672	3156	1748	1296	8300	4650	2590	1435	1932
(3^221^2)	−363	299	1818	1044	786	4490	2504	1495	859	1126
(3^22^2)	902	−637	1527	626	474	2420	1442	862	517	652
(3^31)	−282	123	−330	120	370	1770	1074	651	393	504
(41^6)	−122	104	−273	48	65	13290	7155	3840	2055	2795
(421^4)	151	−116	228	−45	−30	55	3985	2212	1224	1645
(42^21^2)	−744	529	−1090	227	148	−240	1091	1273	732	960
(42^2)	910	−551	1236	−282	−174	243	−1166	1366	441	556
(431^3)	−160	69	−164	42	20	−34	178	−248	76	749
(4321)	502	−410	829	−162	−122	168	−777	814	−112	622
(43^2)	−796	524	−1248	276	210	−198	978	−1164	184	−752
(4^21^4)	116	−83	228	−48	−55	24	−124	156	−20	104
(4^21^2)	−156	129	−240	45	30	−56	266	−284	47	−224
(4^22)	112	−66	152	−29	−22	28	−148	200	−54	112
(51^5)	−135	99	−200	43	26	−48	206	−205	26	−140
(521^3)	542	−355	755	−182	−98	163	−735	774	−104	514
(52^21)	−446	217	−510	162	60	−99	476	−554	84	−323
(531^2)	−332	259	−543	120	78	−103	477	−493	58	−389
(532)	280	−130	331	−122	−41	50	−250	298	−36	189
(541)	181	−140	277	−61	−35	58	−280	299	−40	245
(5^2)	−45	25	−55	20	5	−10	50	−55	5	−45
(61^4)	110	−84	175	−33	−26	41	−176	180	−26	115
(621^2)	−328	221	−498	102	78	−100	450	−512	86	−307
(62^2)	128	−52	128	−40	−16	26	−132	176	−48	80
(631)	169	−144	336	−57	−65	52	−240	279	−40	201
(64)	−72	52	−120	21	22	−20	104	−132	30	−88
(71^3)	−94	69	−142	30	20	−34	146	−147	18	−97
(721)	193	−115	277	−67	−44	51	−239	273	−36	158
(73)	−82	53	−144	36	31	−17	85	−102	10	−68
(81^2)	73	−55	107	−23	−13	27	−118	119	−18	83
(82)	−88	37	−92	31	10	−18	90	−116	26	−56
(91)	−49	43	−87	15	13	−19	86	−87	10	−67
(10)	40	−25	60	−15	−10	10	−50	60	−10	40

Table 4.10 (*continued*)

$w = 10$ (iii)	$h_1h_2h_3h_4$	$h_1h_3^2$	$h_1^2h_2^2$	$h_1^2h_3$	$h_1h_2^3$	$h_1h_2h_3^2$	$h_1h_2^2h_3$	$h_1^2h_2^2$	$h_1^2h_3^2$	$h_1^3h_4$	h_2^5
(1^{10})	12600	4200	6300	3150	30240	15120	7560	5040	2520	1260	252
(21^8)	7700	2660	3990	2030	18480	9408	4788	3248	1652	854	182
(2^21^6)	4700	1690	2520	1310	11100	5778	3006	2078	1080	578	132
(2^31^4)	2865	1077	1584	846	6570	3510	1875	1320	705	390	96
(2^41^2)	1744	688	990	547	3840	2112	1164	832	460	262	70
(2^5)	1060	440	615	355	2220	1260	720	520	300	175	51
(31^7)	3675	1330	2030	1050	9240	4788	2478	1722	889	483	112
(321^5)	2240	850	1280	680	5380	2872	1530	1089	578	327	82
(32^21^3)	1363	545	800	440	3100	1710	942	684	376	220	60
(32^31)	828	350	496	285	1770	1011	579	426	245	147	44
(3^21^4)	1066	432	652	356	2460	1366	754	560	306	186	52
(3^21^2)	647	279	402	230	1390	802	461	348	199	124	38
(3^31)	392	180	246	149	780	468	282	214	130	82	28
	306	145	198	120	600	366	222	174	105	69	24
(41^6)	1490	570	890	470	4050	2139	1128	803	422	242	62
(421^4)	907	367	560	306	2275	1249	684	501	273	164	46
(42^21^2)	551	237	348	199	1270	726	415	310	177	110	34
(42^3)	334	153	214	130	705	420	252	190	115	73	25
(431^3)	431	189	286	162	970	564	325	252	143	94	30
(4321)	**261**	123	174	105	535	325	197	154	93	62	22
	1157										
(43^2)		**65**	84	55	220	143	93	75	49	34	14
	−195	**85**									
(4^21^3)	249	−27	**126**	75	350	218	134	110	66	48	18
			98								
(4^22)	−180	22	−49	**49**	190	124	81	66	43	31	13
				54							
(51^5)	166	−20	44	−20	**1546**	841	456	336	181	111	32
					46						
(521^3)	−648	80	−168	80	−155	**480**	273	207	117	75	24
						561					
(52^21)	465	−57	109	−58	95	−379	**164**	126	76	50	18
							311				
(531^3)	470	−63	136	−58	95	−358	232	**102**	61	43	16
								282			
(532)	−295	41	−59	28	−50	215	−193	−154	**40**	28	12
									157		
(541)	−280	32	−108	44	−50	201	−141	−172	90	**22**	10
										143	
(5^2)	55	−5	20	−5	10	−45	40	40	−35	−35	**6**
(61^4)	−146	20	−34	20	−40	130	−75	−75	35	35	**15**
(621^2)	431	−63	95	−66	96	−339	215	205	−105	−102	−5
(62^3)	−128	16	−31	30	−24	96	−80	−50	36	32	15
(631)	−291	53	−63	44	−48	177	−102	−141	61	73	−5
(64)	132	−22	39	−34	16	−64	42	54	−20	−44	−10
											5
(71^3)	122	−17	28	−14	34	−112	69	63	−35	−29	5
(721)	−247	41	−43	28	−51	188	−136	−107	77	49	−10
(73)	123	−31	15	−10	17	−68	51	41	−41	−20	5
(81^2)	−94	10	−28	14	−27	91	−55	−56	28	29	−5
(82)	92	−10	19	−18	18	−72	62	38	−36	−20	5
(91)	78	−10	24	−10	19	−67	39	48	−20	−29	5
(10)	−60	10	−15	10	−10	40	−30	−30	20	20	−5

Table 4.10 (*continued*)

$w = 10$ (iv)	$h_6h_1^4$	$h_5h_2h_1^3$	$h_4h_2^2$	$h_4h_3h_1$	h_4h_1	$h_7h_1^3$	$h_7h_2h_1$	h_7h_3	$h_6h_1^2$	h_6h_2	h_5h_2	h_{10}
(1^{10})	5040	2520	1260	840	210	720	360	120	90	45	10	1
(21^8)	3360	1708	868	588	154	528	268	92	73	37	9	1
(2^21^6)	2190	1138	591	408	113	378	196	70	58	30	8	1
(2^31^4)	1398	747	399	281	83	264	141	53	45	24	7	1
(2^41^2)	876	484	268	192	61	180	100	40	34	19	6	1
(2^5)	540	310	180	130	45	120	70	30	25	15	5	1
(31^7)	1960	1008	518	357	98	358	183	64	57	29	8	1
(321^5)	1230	652	345	244	72	247	130	48	44	23	7	1
(32^21^3)	758	417	229	166	53	166	91	36	33	18	6	1
(32^31)	460	264	152	112	39	109	63	27	24	14	5	1
(3^21^4)	646	354	193	142	46	152	82	32	32	17	6	1
(3^221^2)	386	222	127	96	34	98	56	24	23	13	5	1
(3^22^2)	228	138	84	64	25	62	38	18	16	10	4	1
(3^21)	186	114	69	55	22	54	33	16	15	9	4	1
(41^6)	1045	544	283	198	57	229	118	42	43	22	7	1
(421^4)	627	340	184	133	42	151	81	31	32	17	6	1
(42^21^2)	371	211	120	89	31	97	55	23	23	13	5	1
(42^3)	217	130	79	59	23	61	37	17	16	10	4	1
(431^3)	303	173	98	75	27	86	48	20	22	12	5	1
(4321)	175	106	64	50	20	53	32	15	15	9	4	1
(43^2)	79	52	34	28	13	27	18	10	9	6	3	1
(4^21^2)	128	78	47	38	16	44	26	12	14	8	4	1
(4^22)	72	47	31	25	12	26	17	9	9	6	3	1
(51^5)	501	266	141	101	31	136	71	26	31	16	6	1
(521^3)	287	161	90	67	23	85	47	19	22	12	5	1
(52^21)	163	97	58	44	17	52	31	14	15	9	4	1
(531^2)	127	77	46	37	15	44	26	12	14	8	4	1
(532)	71	46	30	24	11	26	17	9	9	6	3	1
(541)	48	32	21	18	9	20	13	7	8	5	3	1
(5^2)	16	12	9	8	5	8	6	4	4	3	2	1
(61^4)	209 / 40	115	63	47	16	73	39	15	21	11	5	1
(621^2)	−96	68 / 261	40	31	12	43	25	11	14	8	4	1
(62^2)	24	−78	26 / 44	20	9	25	16	8	9	6	3	1
(631)	48	−141	32	17 / 112	8	20	13	7	8	5	3	1
(64)	−16	54	−22	−44	5 / 34	8	6	4	4	3	2	1
(71^3)	−34	78	−18	−36	10	34 / 34	19	8	13	7	4	1
(721)	51	−137	36	70	−20	−51	12 / 100	6	8	5	3	1
(73)	−17	51	−10	−41	10	17	−41	4 / 31	4	3	2	1
(81^2)	27	−64	18	29	−10	−37	37	−10	7 / 27	4	3	1
(82)	−18	54	−26	−20	10	18	−36	10	−18	3 / 26	2	1
(91)	−19	48	−10	−29	10	19	−29	10	−19	10	2 / 19	1
(10)	10	−30	10	20	−10	−10	20	−10	10	−10	−10	1 / 10

Table 4.11

$w-11$ (i)	h_1^{11}	$h_2h_1^9$	$h_2^2h_1^7$	$h_2^3h_1^5$	$h_2^4h_1^3$	$h_2^5h_1$	$h_3h_1^8$	$h_3h_2h_1^6$	$h_3h_2^2h_1^4$	$h_3h_2^3h_1^2$
(1^{11})	39916800	19958400	9979200	4989600	2494800	1247400	6652800	3326400	1663200	831600
	1									
(21^9)		10160640	5171040	2630880	1338120	680400	3507840	1784160	907200	461160
	−10	101								
(2^21^7)			2678760	1387260	718200	371700	1844640	955080	494340	255780
	36	−368	1361							
(2^31^5)				731520	385740	203400	967680	510300	269100	141900
	−56	581	−2190	3614						
(2^41^3)					207324	111510	506520	272160	146340	78744
	35	−3 5	1430	−2444	1742					
(2^51)						61260	264600	144900	79500	43710
	−6	65	−260	468	−364	91				
(31^8)							1303680	682080	356160	185640
	9	−91	331	−519	325	−55	84			
(321^6)								363840	193740	102990
	−56	574	−2123	3398	−2182	380	−530	3411		
(32^21^4)									105284	57156
	105	−1095	4140	−6818	4544	−828	1005	−6634	13359	
(32^31^2)										31731
	−60	640	−2494	4284	−3035	602	−576	3926	−8298	5583
(32^4)	5	−55	224	−412	329	−85	47	−332	744	−562
(3^31^5)	21	−210	799	−1270	798	−132	206	−1332	2580	−1488
(3^221^3)	−60	630	−2392	3929	−2566	438	−598	3904	−8085	4926
(3^22^21)	30	−324	1278	−2214	1557	−282	300	−2091	4527	−3095
(3^31^2)	10	−100	405	−663	420	−66	105	−714	1458	−856
(3^32)	−4	44	−177	312	−219	34	−42	303	−684	487
(41^7)	−8	81	−295	464	−294	52	−75	471	−888	510
(421^5)	42	−432	1604	−2584	1688	−312	400	−2569	4980	−2961
(42^21^3)	−60	630	−2402	4007	−2745	546	−578	3824	−7717	4845
(42^31)	20	−216	856	−1510	1128	−260	192	−1319	2820	−1954
(431^5)	−30	310	−1152	1843	−1178	207	−298	1925	−3713	2151
(4321^3)	60	−636	2442	−4074	2742	−516	606	−4074	8296	−5118
(432^21)	−12	132	−534	963	−734	170	−120	849	−1879	1352
(43^21)	−12	129	−501	837	−547	94	−129	891	−1849	1103
(4^21^3)	10	−104	389	−628	411	−78	101	−652	1250	−729
(4^221)	−12	129	−504	862	−608	130	−123	836	−1722	1088
(4^23)	3	−33	132	−230	162	−34	33	−234	502	−316
(51^4)	7	−71	259	−408	258	−44	66	−415	787	−458
(521^2)	−30	310	−1157	1875	−1229	219	−288	1860	−3647	2220
(52^21)	30	−318	1227	−2079	1449	−282	291	−1950	4027	−2645
(52^2)	−4	44	−179	329	−264	70	−38	266	−589	442
(531^2)	20	−208	778	−1252	800	−132	202	−1316	2571	−1526
(5321)	−24	258	−1008	1718	−1178	204	−246	1690	−3568	2354
(53^2)	3	−33	132	−229	155	−22	33	−237	525	−355
(541^2)	−12	126	−476	776	−511	94	−124	810	−1576	943
(542)	6	−66	266	−476	360	−84	62	−434	940	−600
(5^21)	3	−32	123	−204	135	−22	32	−214	430	−270
(61^2)	−6	61	−223	352	−223	39	−57	359	−682	393
(621^2)	20	−208	782	−1278	846	−156	194	−1263	2500	−1522
(62^21)	−12	129	−507	881	−636	130	−117	799	−1699	1164
(631^2)	−12	126	−476	773	−497	86	−124	819	−1619	949
(632)	6	−66	266	−474	348	−68	62	−440	978	−704
(641)	6	−64	246	−408	272	−52	64	−428	852	−505
(65)	−2	22	−88	154	−110	21	−22	154	−330	225
(71^2)	5	−51	187	−296	188	−33	48	−303	577	−333
(721^2)	−12	126	−479	793	−533	102	−118	778	−1563	961
(72^2)	3	−33	134	−245	194	−50	29	−203	449	−322
(731)	6	−64	246	−406	262	−44	64	−434	883	−520
(74)	−2	22	−88	154	−111	25	−22	154	−326	208
(81^2)	−4	41	−151	240	−153	26	−39	247	−472	277
(821)	6	−64	248	−420	288	−52	60	−405	842	−545
(83)	−2	22	−88	153	−105	17	−22	157	−345	230
(91^2)	3	−31	115	−184	119	−22	30	−191	364	−213
(92)	−2	22	−89	161	−124	29	−20	140	−309	227
$(10, 1)$	−2	21	−79	127	−80	13	−21	137	−265	150
(11)	1	−11	44	−77	55	−11	11	−77	165	−110

Table 4.11 (*continued*)

$w = 11$ (ii)	$h_1h_2^4$	$h_2^2h_3^3$	$h_1^2h_2h_3^2$	$h_2^3h_3^2h_1$	$h_2^3h_1^3$	$h_2^2h_2$	$h_4h_1^7$	$h_4h_2h_1^5$	$h_4h_2^2h_1^3$	$h_4h_2^3h_1$
(1^{11})	415800	1108800	554400	277200	184800	92400	1663200	831600	415800	207900
(21^9)	234360	614880	312480	158760	107520	54600	922320	468720	238140	120960
(2^21^7)	132300	340200	175980	91000	62580	32340	507780	262710	135870	70245
(2^31^5)	74820	187740	99000	52200	36420	19200	277830	146520	77265	40740
(2^41^3)	42402	103320	55620	29964	21180	11424	151200	81360	43812	23610
(2^51)	24090	56700	31200	17210	12300	6810	81900	45000	24780	13680
(31^8)	96600	249760	129920	67480	47040	24360	381360	198240	102900	53340
(321^6)	54660	137720	73100	38740	27420	14490	207060	109950	58290	30855
(32^21^4)	30996	75700	41068	22256	15972	8640	111930	60750	32939	17841
(32^31^2)	17622	41480	23032	12794	9288	5163	60270	33450	18573	10317
(32^4)	10050	22660	12892	7358	5388	3090	32340	18360	10452	5970
	85									
(3^21^5)		55480	30340	16540	12060	6540	82180	45110	24670	13445
	117	546								
(3^221^3)			17010	9516	7026	3918	44030	24740	13873	7762
	−402	−1638	5235							
(3^23^21)				5478	4080	2352	23520	13530	7788	4485
	272	828	−2872	1923						
(3^31^3)					3098	1792	17010	9930	5787	3363
	62	315	−1043	540	245					
(3^32)						1081	9030	5400	3237	1944
	−34	−126	472	−357	−98	85				
(41^7)							129990	69405	36960	19635
	−44	−182	517	−255	−87	33	70			
(421^5)								38010	20760	11310
	270	999	−2934	1503	504	−198	−373	2044		
(42^21^3)									11642	6519
	−486	−1478	4561	−2483	−795	331	539	−3063	4843	
(42^21)										3762
	240	489	−1508	974	262	−124	−183	1092	−1870	840
(431^4)	−183	−793	2318	−1145	−428	158	277	−1532	2278	−778
(4321^3)	476	1676	−5305	2820	1041	−408	−552	3194	−5096	1896
(432^2)	−170	−327	1126	−732	−198	102	111	−682	1208	−560
(43^21)	−90	−400	1341	−675	−327	111	111	−662	1085	−380
(4^21^3)	68	271	−761	365	131	−44	−100	560	−839	300
(4^221)	−118	−344	1076	−552	−204	66	120	−716	1192	−488
(4^23)	34	106	−304	180	92	−23	−30	188	−332	136
(51^4)	38	159	−457	237	75	−33	−62	327	−467	155
(521^3)	−105	−717	2140	−1177	−354	170	270	−1471	2185	−764
(52^21^2)	262	740	−2348	1425	384	−213	−273	1553	−2457	946
(52^2)	−70	−93	308	−200	−44	22	38	−232	414	−204
(531^2)	120	544	−1616	868	287	−136	−188	1037	−1524	504
(5321)	−196	−686	2276	−1446	−414	258	222	−1290	2058	−752
(53^21)	22	105	−385	267	88	−65	−27	162	−265	88
(541^3)	−86	−340	967	−498	−159	66	124	−698	1041	−365
(542)	84	172	−560	343	90	−43	−62	380	−660	298
(5^21)	21	91	−270	161	42	−26	−32	182	−270	89
(61^4)	−33	−138	402	−202	−70	28	54	−285	412	−140
(621^2)	136	480	−1403	808	263	−118	−183	1006	−1528	556
(62^21)	−118	−298	986	−642	−162	102	111	−644	1060	−444
(631^2)	−78	−347	1062	−540	−210	84	115	−646	980	−337
(632)	68	174	−615	447	107	−85	−56	336	−570	242
(641)	46	186	−557	270	109	−37	−64	372	−583	214
(65)	−21	−66	215	−141	−37	26	22	−132	215	−84
(71^4)	29	117	−342	169	61	−22	−46	243	−352	118
(721^3)	−94	−303	947	−498	−177	66	112	−625	972	−357
(72^2)	50	72	−243	155	39	−17	−29	176	−314	154
(731)	42	190	−616	306	130	−46	−58	335	−528	178
(74)	−25	−68	222	−108	−50	11	22	−136	236	−100
(81^3)	−22	−96	279	−148	−47	22	38	−201	289	−97
(821)	46	157	−515	313	93	−52	−57	324	−515	192
(83)	−17	−69	250	−162	−58	35	19	−114	190	−68
(91^2)	20	76	−215	108	35	−14	−30	161	−233	81
(92)	−29	−51	173	−120	−25	17	20	−120	209	−98
$(10, 1)$	−11	−58	170	−81	−32	11	21	−116	170	−54
(11)	11	33	−110	66	22	−11	−11	66	−110	44

Table 4.11 (*continued*)

$\omega = 11$ (iii)	$h_4h_3h_1^4$	$h_4h_3h_2h_1^3$	$h_4h_3h_2^2$	$h_4h_2^2h_1$	$h_4^2h_1^3$	$h_4^2h_2h_1$	$h_4^2h_3$	$h_3h_1^6$	$h_3h_2h_1^4$	$h_3h_2^2h_1^3$	$h_3h_2^3$	$h_3^2h_1^5$
(1^{11})	277200	138600	69300	46200	69300	34650	11550	332640	166320	83160	41580	55440
(21^9)	161280	81900	41580	28140	42210	21420	7350	196560	99792	50652	25704	34272
(2^21^7)	93450	48300	24955	17150	25620	13230	4690	114660	59262	30618	15813	21042
(2^31^5)	53910	28425	14985	10455	15480	8160	3000	66150	34884	18393	9696	12834
(2^41^3)	30960	16692	9006	6372	9306	5025	1923	37800	20376	10992	5934	7776
(2^51)	17700	9780	5420	3880	5565	3090	1235	21420	11820	6540	3630	4680
(31^8)	71680	37100	19180	13300	20230	10430	3710	92400	47712	24612	12684	17024
(321^6)	41270	21815	11515	8120	12230	6440	2380	52620	27708	14664	7725	10334
(32^21^4)	23642	12797	6919	4957	7344	3966	1530	29730	16094	8703	4701	6234
(32^31^2)	13478	7489	4163	3023	4382	2437	985	16680	9264	5148	2862	3736
(32^4)	7648	4372	2510	1840	2599	1495	635	9300	5304	3036	1746	2224
(3^21^5)	18050	9810	5315	3860	5820	3140	1220	23060	12574	6830	3697	4978
(3^221^3)	10254	5733	3197	2359	3458	1926	788	12830	7192	4021	2242	2972
(3^231)	5798	3344	1927	1438	2042	1179	509	7100	4094	2362	1363	1762
(3^22)	4386	2553	1479	1127	1602	930	410	5370	3138	1827	1059	1392
(41^7)	2460	1482	894	685	936	567	265	2940	1770	1068	645	816
(421^5)	26635	14105	7455	5320	8260	4340	1610	37590	19761	10374	5439	7343
(42^21^3)	15215	8265	4480	3255	4970	2680	1040	20805	11256	6078	3276	4397
(42^31)	8642	4830	2696	1990	2964	1649	673	11460	6386	3555	1977	2616
(431^4)	4883	2815	1626	1214	1754	1012	436	6285	3609	2076	1197	1546
	6573	3697	2070	1555	2356	1310	540	8615	4871	2738	1531	2069
	1237											
(4321^3)	−2551	**2151**	1248	951	1384	801	351	4700	2742	1505	925	1218
		5904										
(432^2)	509	−1320	**755**	579	808	489	228	2555	1538	928	562	712
			426									
(43^21)	576	−1464	294	**456**	630	385	185	1890	1159	709	432	557
				488								
(4^31^2)	−465	920	−188	−188	**950**	549	243	2990	1782	1052	616	834
					200							
(4^221)	578	−1360	325	320	−240	**334**	159	1610	992	609	373	484
						383						
(4^23)	−166	456	−111	−161	60	−119	**85**	630	410	267	174	216
							69					
(51^6)	−239	472	−95	−91	84	−96	22	**13326**	7185	3864	2073	2819
								59				
(521^4)	1090	−2249	485	440	−391	470	−110	−256	**4010**	2232	1239	1665
									1160			
(52^21^2)	−1135	2532	−624	−495	413	−552	132	259	−1240	**1289**	744	976
										1447		
(52^3)	157	−392	134	66	−65	109	−25	−34	175	−238	**450**	568
											70	
(531^5)	−845	1708	−340	−360	316	−360	88	176	−815	839	−106	**765**
(5321)	1024	−2363	550	529	−359	484	−130	−214	1060	−1257	166	**651**
(53^2)	−140	357	−72	−111	41	−60	23	27	−140	174	−16	−826
(541^4)	593	−1151	239	217	−202	288	−61	−112	535	−578	86	121
(542)	−300	724	−214	−137	131	−210	50	54	−280	362	−84	−452
(5^21)	−160	307	−63	−53	72	−68	11	32	−160	181	−21	214
(61^5)	204	−417	85	86	−69	86	−22	−52	221	−219	29	144
(621^3)	−731	1576	−352	−330	254	−336	88	175	−784	828	−118	−146
(62^21)	452	−1060	296	206	−163	235	−59	−107	516	−612	100	533
(631^4)	529	−1140	231	271	−191	250	−73	−107	492	−510	66	−342
(632)	−255	630	−192	−129	87	−126	34	56	−285	354	−50	−303
(641)	−321	684	−138	−159	141	−186	50	56	−269	291	−46	216
(65)	115	−252	63	48	−52	63	−11	−22	115	−141	21	230
(71^5)	−174	363	−73	−80	57	−74	22	45	−191	192	−27	−104
(721^3)	453	−1037	233	244	−150	222	−73	−108	489	−538	88	122
(72^21)	−117	302	−104	−61	45	−79	25	27	−135	178	−50	−317
(731)	−281	675	−132	−201	91	−144	62	54	−253	273	−36	76
(74)	118	−300	75	89	−50	89	−39	−18	90	−108	25	199
(81^3)	146	−293	59	59	−50	60	−15	−38	163	−164	20	−72
(821)	−233	548	−126	−124	76	−114	34	57	−267	305	−40	−108
(83)	95	−252	57	81	−26	45	−23	−19	95	−114	11	176
(91^2)	−122	237	−51	−43	46	−52	11	30	−131	132	−20	−76
(92)	82	−204	69	33	−31	51	−11	−20	100	−129	29	92
$(10, 1)$	95	−192	33	43	−37	43	−11	−21	95	−96	11	−62
(11)	−55	132	−33	−33	22	−33	11	11	−55	66	−11	−74
												44

Table 4.11 (*continued*)

$w = 11$ (iv)	$h_6h_5h_3h_1$	$h_5h_3^2$	$h_5h_4h_1^2$	$h_4h_4h_2$	$h_4^2h_1$	$h_4h_1^6$	$h_4h_3h_1^3$	$h_4h_3^2h_1$	$h_4h_2h_1^5$	$h_4h_2^3$	$h_3h_4h_1$	h_3h_6
(1^{11})	27720	9240	13860	6930	2772	55440	27720	13860	9240	4620	2310	462
(21^9)	17388	5964	8946	4536	1890	35280	17892	9072	6132	3108	1596	336
(2^21^7)	10864	3850	5754	2968	1288	22050	11382	5873	4032	2079	1099	245
(2^31^5)	6765	2487	3684	1941	876	13560	7149	3768	2629	1385	754	179
(2^41^3)	4200	1608	2346	1269	594	8220	4440	2400	1700	920	515	131
(2^51)	2600	1040	1485	830	401	4920	2730	1520	1090	610	350	96
(31^8)	8764	3108	4718	2422	1078	19040	9772	5012	3444	1764	938	210
(321^6)	5437	2006	3023	1585	736	11530	6058	3179	2229	1167	643	154
(32^21^4)	3365	1297	1924	1036	500	6890	3719	2005	1433	771	439	113
(32^31^2)	2078	840	1216	677	338	4070	2263	1259	914	509	298	83
(32^4)	1280	544	763	443	227	2380	1366	788	578	336	201	61
(3^21^5)	2686	1044	1584	848	424	5690	3074	1655	1198	641	374	98
(3^221^3)	1655	677	998	553	286	3320	1854	1032	762	422	254	72
(3^231)	1018	440	624	361	192	1920	1110	642	480	278	171	53
(3^23^2)	807	355	510	294	162	1530	894	519	398	229	146	46
(31^7)	495	232	315	192	108	870	528	321	247	151	97	34
(421^5)	3843	1414	2177	1134	546	9275	4816	2499	1743	903	497	119
(42^21^3)	2365	913	1389	743	374	5410	2896	1548	1107	590	339	88
(42^31)	1453	591	878	486	254	3125	1730	957	699	386	230	65
(431^4)	891	383	550	318	171	1790	1027	591	438	253	155	48
(4321^2)	1151	475	726	399	218	2485	1386	769	575	316	196	57
(432^2)	705	309	452	260	146	1410	818	473	360	207	132	42
(43^21)	431	201	279	170	97	795	480	291	223	136	88	31
(4^21^4)	340	163	227	138	82	615	378	231	183	111	75	27
(4^221^2)	484	216	332	189	114	990	580	337	264	151	102	34
(4^231)	295	141	203	123	75	550	337	206	163	99	68	25
(4^23^2)	141	75	99	65	41	230	151	99	81	53	38	16
(51^7)	1508	582	908	482	254	4051	2140	1129	804	423	243	63
(521^5)	922	377	575	316	174	2276	1250	685	502	274	165	47
(52^21^3)	563	245	360	207	118	1271	727	416	311	178	111	35
(52^3)	343	159	223	136	79	706	421	253	191	116	74	26
(531^3)	443	197	298	170	102	971	565	326	253	144	95	31
(5321)	270	129	183	111	68	536	326	198	155	94	63	23
	1305											
(53^2)	−228	69	90	59	38	221	144	94	76	50	35	15
		65										
(541^3)	534	−63	135	81	54	351	219	135	111	67	49	19
			390									
(542)	−326	37	−183	53	35	191	125	82	67	44	32	14
				160								
(5^21)	−191	26	−128	52	26	112	76	51	44	29	23	11
					56							
(61^5)	174	−22	87	−44	−22	1546	841	456	336	181	111	32
						51						
(621^3)	−685	91	−331	182	89	−170	480	273	207	117	75	24
							608					
(62^21)	518	−72	223	−146	−68	102	−402	164	126	76	50	18
								326				
(631^2)	485	−72	260	−133	−73	102	−378	235	102	61	43	16
									293			
(632)	−366	61	−135	92	52	−51	216	−192	−147	40	28	12
										152		
(641)	−270	37	−198	102	57	−51	200	−132	−168	74	22	10
											136	
(65)	156	−26	93	−52	−41	17	−74	63	63	−52	−52	6
												41
(71^4)	−150	19	−69	38	16	−45	146	−84	−84	39	39	−11
(721^3)	429	−57	185	−121	−43	108	−381	241	230	−117	−114	33
(72^21)	−116	11	−56	54	11	−27	108	−90	−86	40	36	−11
(731)	−273	46	−114	72	27	−54	199	−114	−159	68	82	−22
(74)	94	−11	61	−50	−11	18	−72	47	61	−22	−50	11
(81^3)	136	−19	62	−31	−16	38	−125	77	70	−39	−32	11
(821)	−276	46	−95	62	27	−57	210	−152	−119	86	54	−22
(83)	138	−35	33	−22	−11	19	−76	57	57	−46	−22	11
(91^2)	−104	11	−62	31	16	−30	101	−61	−62	31	32	−11
(92)	102	−11	42	−40	−11	20	−80	69	42	−40	−22	11
$(10, 1)$	86	−11	53	−22	−16	21	−74	43	53	−22	−32	11
(11)	−66	11	−33	22	11	−11	44	−33	−33	22	22	−11

Table 4.11 (*continued*)

$w=11$ (v)	$h_7h_1^4$	$h_7h_3h_1^2$	$h_7h_2^2$	$h_7h_2h_1$	h_7h_4	$h_6h_1^5$	$h_6h_3h_1$	h_6h_2	$h_5h_1^6$	h_5h_3	$h_5h_2h_1^2$	$h_4h_3h_1^3$	$h_4h_2^2h_1$	$h_3^3h_1^2$	$h_3h_2^3$	h_{11}	$h_{10}h_1$	h_9h_2	h_8h_3	$h_8h_2h_1$	$h_7h_4h_1$	$h_7h_3h_2$	h_6h_5	$h_6h_4h_1^2$	$h_6h_3h_1^3$	$h_6h_2^2h_1$	$h_5^2h_1^2$	$h_5h_4h_1^3$	$h_5h_3h_2h_1$	$h_5h_2^3$	$h_4^2h_3h_1^2$	$h_4h_3^2h_2$	$h_4h_2^3h_1$	$h_3^4h_1$	$h_3^3h_2^2$	$h_3^2h_2^3$	h_{12}
(1^{11})	7920	3960	1980	1320	330	990	495	165	110	55	11	1																								
(21^9)	5472	2772	1404	948	246	747	378	129	91	46	10	1																								
(2^21^7)	3702	1908	983	674	183	552	284	100	74	38	9	1																								
(2^31^5)	2454	1293	681	475	136	399	210	77	59	31	8	1																								
(2^41^3)	1596	864	468	332	101	282	153	59	46	25	7	1																								
(2^51)	1020	570	320	230	75	195	110	45	35	20	6	1																								
(31^8)	3392	1732	884	604	162	529	269	93	73	37	9	1																								
(321^6)	2218	1159	605	422	120	370	197	71	58	30	8	1																								
(32^21^4)	1422	765	411	293	89	265	142	54	45	24	7	1																								
(32^31^2)	896	499	278	202	66	181	101	41	34	19	6	1																								
(32^4)	556	322	188	138	49	121	71	31	25	15	5	1																								
(3^21^5)	1254	670	357	256	78	248	131	49	44	23	7	1																								
(3^221^3)	778	432	230	176	58	167	92	37	33	18	6	1																								
(3^22^21)	476	276	160	120	43	110	64	28	24	14	5	1																								
(3^32^2)	402	234	135	104	38	99	57	25	23	13	5	1																								
(41^7)	240	147	90	70	28	63	39	19	16	10	4	1																								
(421^5)	1901	1009	519	358	99	358	183	64	57	29	8	1																								
(42^21^3)	1231	653	346	245	73	247	130	48	44	23	7	1																								
(42^31)	759	418	230	167	54	166	91	36	33	18	6	1																								
(4^221)	461	265	153	113	40	109	63	27	24	14	5	1																								
(431^4)	647	355	194	143	47	152	82	32	32	17	6	1																								
(4321^2)	387	223	128	97	35	98	56	24	23	13	5	1																								
(432^2)	229	139	85	65	26	62	38	18	16	10	4	1																								
(43^21)	187	115	70	56	23	54	33	16	15	9	4	1																								
(4^31^3)	304	174	99	76	28	86	48	20	22	12	5	1																								
(4^321)	176	107	65	51	21	53	32	15	15	9	4	1																								
(4^23)	80	53	35	29	14	27	18	10	9	6	3	1																								
(51^6)	1045	544	283	198	57	229	118	42	43	22	7	1																								
(521^4)	627	340	184	133	42	151	81	31	32	17	6	1																								
(52^21^2)	371	211	120	89	31	97	55	23	23	13	5	1																								
(52^2)	217	130	79	59	23	61	37	17	16	10	4	1																								
(531^3)	303	173	98	75	27	86	48	20	22	12	5	1																								
(5321)	175	106	64	50	20	53	32	15	15	9	4	1																								
(53^2)	79	52	34	28	13	27	18	10	9	6	3	1																								
(541^2)	128	78	47	38	16	44	26	12	14	8	4	1																								
(542)	72	47	31	25	12	26	17	9	9	6	3	1																								
(5^21)	48	32	21	18	9	20	13	7	8	5	3	1																								
(61^5)	501	266	141	101	31	136	71	26	31	16	6	1																								
(621^3)	287	161	90	67	23	85	47	19	22	12	5	1																								
(62^21)	163	97	58	44	17	52	31	14	15	9	4	1																								
(631^2)	127	77	46	37	15	44	26	12	14	8	4	1																								
(632)	71	46	30	24	11	26	17	9	9	6	3	1																								
(641)	48	32	21	18	9	20	13	7	8	5	3	1																								
(65)	16	12	9	8	5	8	6	4	4	3	2	1																								
(71^4)	209/45	115	63	47	16	73	39	15	21	11	5	1																								
(721^2)	−108	68/294	40	31	12	43	25	11	14	8	4	1																								
(72^2)	27	−88	26/50	20	9	25	16	8	9	6	3	1																								
(731)	54	−159	36	17/127	8	20	13	7	8	5	3	1																								
(74)	−18	61	−25	−50	5/39	8	6	4	4	3	2	1																								
(81^3)	−38	87	−20	−40	11	34/38	19	8	13	7	4	1																								
(821)	57	−153	40	78	−22	−57	12/112	6	8	5	3	1																								
(83)	−19	57	−11	−46	11	19	−46	4/35	4	3	2	1																								
(91^2)	30	−71	20	32	−11	−30	41	−11	7/30	4	3	1																								
(92)	−20	60	−29	−22	11	20	−40	11	−20	3/39	2	1																								
$(10,1)$	−21	53	−11	−32	11	21	−32	11	−21	11	2/21	1																								
(11)	11	−33	11	22	−11	−11	22	−11	11	−11	−11	1/11																								

Table 4.12

$w-12$ (i)	h_1^{12}	$h_2h_1^{10}$	$h_3^2h_1^6$	$h_2^3h_1^6$	$h_2^4h_1^4$	$h_3^3h_1^3$	h_2^6	$h_2h_3^2h_1^3$	$h_2h_3^3h_1^3$	$h_2h_3^4h_1^3$
(1^{12})	479001600	239500800	119750400	59875200	29937600	14968800	7484400	79833600	39916800	19958400
	1									
(21^{10})		121564800	61689600	31298400	15876000	8051400	4082400	41731200	21168000	10735200
	−11	122								
(2^21^8)			31772160	16359840	8421840	4334400	2230200	21772800	11208960	5769120
	45	−504	2107							
(2^21^9)				8550900	4469040	2335500	1220400	11340000	5927040	3097620
	−84	952	−4039	7890						
(2^41^4)					2372256	1259640	669060	5896800	3129840	1661760
	70	−805	3480	−6970	6376					
(2^31^6)						680070	367560	3061800	1650600	890700
	−21	246	−1090	2260	−2178	812				
(2^6)							202410	1587600	869400	477000
	1	−12	55	−120	126	−56	7			
(31^9)								15240960	7922880	4112640
	10	−111	458	−861	720	−215	10	103		
(321^7)									4184040	2206680
	−72	808	−3378	6452	−5500	1680	−80	−750	5545	
(32^21^5)										1183000
	168	−1911	8124	−15846	13872	−4380	216	1767	−13314	32770
(32^21^3)	−140	1620	−7040	14136	−12872	4288	−224	−1480	11418	−29044
(32^41)	30	−355	1590	−3332	3236	−1197	70	315	−2500	6630
(3^21^6)	28	−315	1317	−2504	2106	−624	28	300	−2225	5330
(3^221^4)	−105	1200	−5115	9958	−8614	2628	−120	−1140	8654	−21363
(3^221^2)	90	−1050	4596	−9264	8385	−2694	126	984	−7698	19830
(3^23^2)	−10	120	−546	1164	−1149	426	−20	−108	876	−2388
(3^31^3)	20	−230	984	−1911	1626	−474	20	226	−1734	4209
(3^321)	−20	236	−1044	2118	−1902	576	−24	−228	1821	−4788
(3^4)	1	−12	54	−111	99	−27	1	12	−99	270
(41^8)	−9	100	−413	778	−655	200	−10	−93	675	−1584
(421^6)	56	−630	2641	−5066	4362	−1372	72	586	−4326	10364
(42^21^4)	−105	1200	−5130	10088	−8972	2952	−168	−1110	8374	−20633
(42^31^2)	60	−700	3074	−6268	5867	−2086	140	636	−4932	12630
(42^4)	−5	60	−274	592	−609	260	−30	−52	416	−1116
(431^5)	−42	474	−1988	3796	−3224	984	−48	−454	3364	−8037
(4321^3)	120	−1380	5924	−11642	10242	−3264	168	1316	−10024	24810
(432^21)	−60	708	−3144	6468	−6060	2100	−120	−660	5214	−13605
(43^21^2)	−30	348	−1503	2952	−2556	780	−36	−345	2667	−6651
(43^22)	12	−144	651	−1362	1287	−434	20	138	−1122	3024
(4^21^4)	15	−170	716	−1375	1183	−375	21	164	−1214	2890
(4^21^22)	−30	348	−1509	3006	−2709	918	−60	−333	2550	−6338
(4^23^2)	6	−72	327	−696	694	−280	30	66	−528	1400
(4^231)	12	−141	618	−1236	1100	−358	20	141	−1104	2786
(4^5)	−1	12	−54	112	−106	40	−4	−12	96	−248
(51^7)	8	−89	368	−694	584	−176	8	83	−603	1420
(521^5)	−42	474	−1994	3840	−3316	1032	−48	−442	3275	−7901
(52^21^3)	60	−690	2972	−5899	5100	−1740	84	638	−4852	12115
(52^31)	−20	236	−1052	2190	−2110	772	−40	−212	1667	−4371
(531^4)	30	−340	1432	−2745	2334	−699	30	328	−2443	5882
(5321^2)	−60	696	−3018	6000	−5334	1680	−72	−666	5136	−12946
(532^2)	12	−144	654	−1389	1364	−502	20	132	−1065	2878
(53^21)	12	−141	618	−1233	1078	−316	12	141	−1113	2857
(541^3)	−20	228	−966	1866	−1612	504	−24	−222	1654	−3972
(5421)	24	−282	1242	−2522	2322	−788	40	270	−2104	5368
(543)	−6	72	−324	670	−620	204	−8	−72	582	−1540
(51^4)	6	−69	295	−575	500	−154	6	68	−512	1246
(5^21)	−3	36	−163	344	−335	122	−4	−34	272	−724
(61^6)	−7	78	−323	610	−514	156	−8	−73	531	−1252
(621^4)	30	−340	1437	−2782	2417	−762	42	318	−2368	5746
(62^21^2)	−30	348	−1515	3048	−2787	942	−60	−321	2469	−6259
(62^3)	4	−48	219	−472	482	−200	20	42	−336	904
(631^3)	−20	228	−966	1862	−1590	480	−24	−222	1666	−4039
(6321)	24	−282	1242	−2516	2284	−732	40	270	−2122	5486
(63^3)	−3	36	−162	334	−303	90	−4	−36	294	−792
(641^2)	12	−138	590	−1150	1003	−322	20	136	−1024	2480
(642)	−6	72	−326	688	−672	256	−24	−68	544	−1440
(651)	−6	70	−304	602	−530	164	−8	−70	538	−1338
(6^2)	1	−12	54	−112	105	−36	3	12	−96	252
(71^5)	6	−67	278	−526	444	−135	6	63	−459	1084
(721^3)	−20	228	−970	1892	−1658	528	−24	−214	1605	−3927
(72^21)	12	−141	624	−1283	1208	−422	20	129	−1009	2621
(731^2)	12	−138	590	−1147	986	−300	12	136	−1033	2532
(732)	−6	72	−326	686	−658	228	−8	−68	550	−1484
(741)	−6	70	−304	602	−532	172	−8	−70	538	−1330
(75)	2	−24	108	−224	210	−71	2	24	−192	504
(81^4)	−5	56	−233	442	−374	114	−6	−53	387	−916
(821^2)	12	−138	593	−1170	1039	−338	20	130	−986	2444
(82^2)	−3	36	−164	352	−356	144	−12	−32	256	−688
(831)	−6	70	−304	600	−520	156	−8	−70	544	−1367
(84)	2	−24	108	−224	211	−76	6	24	−192	500
(91^3)	4	−45	188	−358	304	−92	4	43	−315	748
(921)	−6	70	−306	616	−558	180	−8	−66	511	−1303
(93)	2	−24	108	−223	204	−63	2	24	−195	522
$(10, 1^2)$	−3	34	−143	274	−235	74	−4	−33	243	−577
$(10, 2)$	2	−24	109	−232	230	−88	6	22	−176	472
$(11, 1)$	2	−23	98	−189	160	−47	2	23	−173	417
(12)	−1	12	−54	112	−105	36	−2	−12	96	−252

Table 4.12 (*continued*)

$w=12$ (ii)	$h_6h_1^2h_1^2$	$h_6h_4^2h_1$	$h_6^2h_1^2$	$h_6^2h_2h_1^2h_1^2$	$h_6^2h_2^2h_1^2$	$h_6^2h_3^2$	$h_6^2h_3^2h_1^2$	$h_6^2h_3h_2h_1$	h_6^2	$h_4h_1^2$	$h_4h_2h_1^2$
(1^{12})	9070200	4989600	13305600	6652800	3326400	1663200	2217600	1108800	369600	19958400	9979200
(21^{10})	5243200	2759400	7257600	3679200	1864800	945000	1260000	638400	218400	10886400	5518800
(2^21^8)	2908550	1527120	3951360	2032800	1045520	537600	715680	367920	129360	5906880	3039120
(2^21^8)	1618740	845820	2147040	1121940	586200	306240	406260	212220	76800	3190320	1667250
(2^41^4)	882576	468888	1164240	618480	328656	174696	230400	122496	45696	1716120	911820
(2^31^6)	481140	260190	630000	340500	184240	99810	130500	70740	27240	919800	496800
(2^6)	262260	144540	340200	187200	103260	57120	73800	40860	16260	491400	270000
(31^5)	2131920	1103760	2862720	1481760	766080	395640	530880	273840	97440	4354560	2252880
(321^3)	1162350	611520	1554000	817600	429660	225540	301560	158130	57960	2338560	1230600
(32^21^2)	633600	339120	841680	450540	240960	128760	171120	91380	34560	1251000	670170
(32^21^2)	**345405**	188256	454860	247920	135114	73626	96960	52833	20652	667800	363960
	27013										
(32^41)		**104628**	245280	136220	75744	42174	54840	30552	12360	355320	197160
	−6630	**1889**									
(3^21^6)			**608320**	328240	176720	94940	127200	68220	26160	907200	490560
	−4506	948	**926**								
(3^221^4)				**180516**	99116	54332	72108	39492	15672	482100	265570
	18750	−4098	−3606	**14613**							
$(3^22^21^2)$					**55576**	31150	40788	22866	9408	255640	143440
	−18513	4350	3156	−13482	**13627**						
(3^23^2)						**17896**	23016	13236	5652	135240	77310
	2466	−732	−339	1521	−1702	**316**					
(3^231^2)							**30372**	17130	7168	183120	104010
	−3714	762	762	−3141	2852	−292	**745**				
(3^231)								**9929**	4324	96600	55920
	4509	−974	−783	3492	−3693	402	−808	**1183**			
(3^4)									**2008**	36120	21600
	−261	48	45	−216	252	−21	56	−105	**15**		
(41^8)										**1421280**	753480
	1326	−288	−269	1008	−858	96	−195	189	−9	**87**	
(421^4)											**407415**
	−8900	2000	1731	−6646	5823	−678	1302	−1302	63	−548	**3523**
(42^21^4)	18383	−4350	−3340	13254	−12120	1500	−2613	2745	−135	1038	−6848
(42^21^2)	−11910	3072	1926	−7968	7790	−1066	1534	−1758	87	−600	4093
(42^3)	1136	−340	−153	654	−674	100	−116	136	−6	52	−374
(431^5)	6798	−1473	−1404	5385	−4626	513	−1116	1080	−54	424	−2741
(4321^3)	−21906	5004	4188	−16830	15246	−1804	3582	−3686	192	−1214	8098
(432^21)	13006	−3310	−2118	9053	−9146	1250	−1864	2274	−126	606	−4203
(43^31^2)	5777	−1250	−1188	4888	−4323	474	−1182	1143	−63	309	−2100
(43^2)	−2078	748	477	−2156	2328	−318	486	−678	45	−120	852
(4^21^4)	−2446	546	509	−1911	1605	−181	384	−348	15	−161	1048
(4^221^2)	5638	−1346	−1069	4252	−3756	450	−894	834	−36	324	−2200
(4^22^2)	−1376	380	210	−892	880	−112	176	−192	9	−66	476
(4^231)	−2448	562	500	−2066	1776	−204	516	−432	18	−132	922
(4^3)	224	−56	−44	184	−152	16	−48	32	−1	12	−88
(51^7)	−1198	260	239	−901	786	−90	171	−177	9	−78	488
(521^5)	6875	−1557	−1303	5049	−4569	555	−966	1044	−54	415	−2657
(52^21^3)	−11051	2694	1916	−7721	7381	−998	1473	−1689	87	−599	3952
(52^31)	4316	−1234	−633	2669	−2764	482	−478	574	−24	203	−1395
(531^4)	−5041	1086	1021	−3955	3529	−399	802	−860	48	−307	1979
(5321^3)	11758	−2724	−2138	8799	−8468	1064	−1825	2148	−126	612	−4094
(532^3)	−2946	884	417	−1831	1998	−390	342	−432	18	−123	865
(53^31)	−2577	538	499	−2147	2091	−222	519	−633	45	−123	842
(541^5)	3417	−770	−696	2631	−2299	274	−511	514	−24	220	−1432
(5421)	−4088	1252	874	−3578	3424	−472	716	−786	36	−264	1810
(543)	1460	−360	−262	1150	−1140	156	−280	313	−15	66	−470
(5^212^2)	−1095	250	217	−830	760	−97	155	−174	9	−68	444
(5^12)	730	−220	−109	460	−485	99	−80	93	−3	34	−238
(61^6)	1053	−225	−211	801	−696	75	−156	162	−9	69	−432
(621^4)	−5012	1110	945	−3708	3377	−380	728	−812	48	−300	1931
(62^21^2)	5802	−1394	−971	4009	−3958	484	−776	984	−63	303	−2024
(62^3)	−928	280	123	−530	566	−92	92	−120	6	−42	298
(631^3)	3463	−730	−706	2774	−2481	258	−585	642	−42	208	−1353
(6321)	−5128	1156	884	−3788	3888	−432	798	−1122	90	−246	1674
(63^2)	762	−156	−132	612	−684	66	−151	255	−30	30	−210
(641^4)	−2130	472	440	−1688	1455	−156	348	−342	18	−136	900
(642)	1408	−308	−222	948	−971	118	−186	244	−15	68	−484
(651)	1190	−259	−234	930	−874	95	−188	229	−15	70	−468
(6^3)	−240	54	42	−180	189	−18	38	−60	6	−12	84
(71^5)	−913	201	183	−696	597	−70	138	−132	6	−60	376
(721^3)	3459	−810	−643	2547	−2303	294	−512	520	−24	203	−1317
(72^21)	−2528	674	391	−1654	1688	−260	316	−384	18	−123	836
(731^2)	−2182	484	447	−1784	1560	−186	399	−378	18	−127	839
(732)	1492	−408	−224	1001	−1102	182	−203	283	−15	62	−434
(741)	1157	−271	−238	941	−795	99	−213	174	−6	70	−476
(75)	−485	130	84	−355	355	−59	75	−83	3	−24	168
(81^4)	773	−166	−155	591	−513	56	−117	118	−6	51	−320
(821^2)	−2178	504	397	−1602	1475	−176	328	−348	18	−124	815
(82^2)	704	−216	−95	410	−431	78	−74	84	−3	32	−226
(831)	1195	−247	−242	1010	−917	91	−243	253	−15	64	−433
(84)	−464	116	86	−364	332	−40	88	−80	3	−24	172
(91^3)	−637	138	127	−383	429	−48	93	−102	6	−42	264
(921)	1208	−283	−205	859	−858	105	−172	223	−15	63	−421
(93)	−501	114	87	−396	423	−51	97	−138	12	−21	147
$(10, 1^2)$	490	−111	−100	375	−321	39	−72	69	−3	33	−210
$(10, 2)$	−480	140	67	−290	315	−54	50	−68	3	−22	154
$(11, 1)$	−350	71	75	−290	246	−24	62	−59	3	−23	150
(12)	240	−60	−42	180	−180	24	−40	48	−3	12	−84

Table 4.12 (*continued*)

$w = 12$ (iii)	$h_4h_2^2h_1^4$	$h_4h_2^3h_1^2$	$h_4h_2^4$	$h_4h_3h_1^5$	$h_4h_3h_2h_1^3$	$h_4h_2h_2^2h_1$	$h_4h_3^2h_1^3$	$h_4h_3^2h_2$	$h_4^2h_1^6$	$h_4^2h_2h_1^2$	$h_4^3h_2^2$	$h_4^3h_3h_1$
(1^{12})	4989600	2494800	1247400	3326400	1663200	831600	554400	277200	831600	415800	207900	138600
(21^{10})	2797200	1417500	718200	1890000	957600	485100	327600	165900	491400	248850	126000	85050
(2^21^8)	1563240	803880	413280	1070160	550200	282800	193480	99400	289380	148680	76370	52220
(2^31^6)	871200	455175	237780	603810	315450	164775	114180	59625	169740	88650	46290	32070
(2^41^4)	484308	257400	136842	339480	180468	95964	67308	35808	99144	52740	28062	19692
(2^51^2)	268620	145410	78810	190200	103020	55870	39620	21530	57660	31305	17020	12085
(2^6)	148680	82080	45450	106200	58680	32520	23280	12960	33390	18540	10335	7410
(31^9)	1164240	601020	309960	806400	415800	214200	147840	76020	224280	115290	59220	40740
(321^7)	646800	339570	178080	454020	238105	124740	87290	45640	131460	68740	35910	25060
(32^21^5)	358540	191655	102360	254660	136040	72610	51480	27435	76660	40870	21770	15410
(32^31^3)	198348	108084	58890	142320	77550	42252	30312	16512	44484	24234	13200	9468
(32^41)	109524	60918	33930	79260	44108	24582	17812	9950	25692	14333	8010	5810
(3^21^9)	264600	142380	76440	190840	102490	54920	39400	21030	59380	31720	16910	12080
(3^221^6)	146040	80175	43940	106350	58336	31941	23210	12671	34340	18776	10246	7432
(3^231^4)	80468	45128	25298	59660	33128	18574	13640	7644	19764	11084	6212	4564
(3^232)	44268	25395	14598	32690	18770	10801	7994	4617	11324	6527	3772	2797
(3^31^3)	58980	33375	18840	43920	24858	14028	10455	5877	15180	8568	4818	3594
(3^221)	32388	18762	10866	24240	14061	8154	6126	3554	8664	5034	2922	2202
(3^4)	12948	7776	4674	9840	5928	3576	2740	1660	3744	2268	1374	1060
(41^5)	398580	210420	110880	284760	149940	78820	55860	29260	80310	45150	23590	16590
(421^5)	219810	118350	63600	159195	85510	45850	32970	17615	50310	26860	14320	10230
(42^21^4)	**121027**	66537	36528	88640	48647	26663	19427	10619	29140	15922	8690	6302
(42^31^2)	**13791** −8664	**37398** **5908**	21018	49170	27609	15503	11421	6410	16782	9408	5276	3876
(42^4)	864	−700	**12126** **140**	27180	15632	9014	6696	3874	9615	5543	3209	2379
(431^3)	5308	−3099	264	**66130** **2246**	36560	20145	14890	8155	22570	12370	6760	4960
(4321^2)	−16376	10092	−920	−6622	**20713** 20777	11707	8757	4929	12930	7288	4098	3054
(432^21)	9022	−6134	640	3328	−11202	**6804** **6988**	5133	2983	7372	4282	2488	1875
(43^21^2)	4299	−2582	212	1840	−6019	3120	**3941** **2076**	2299	5630	3302	1928	1482
(43^2)	−1874	1286	−112	−706	2510	−1668	−780	**1395** **498**	3182	1930	1172	906
(4^31^4)	−2040	1215	−116	−872	2521	−1258	−673	244	**8512** **372**	4816	2714	2052
(4^321^2)	4550	−2930	328	1818	−5738	3108	1644	−624	−764	**2819** **1808**	1646	1263
(4^22^2)	−1088	848	−156	−368	1280	−848	−336	156	164	−464	**1002** **207**	774
(4^231)	−1952	1228	−120	−828	2800	−1464	−1024	337	328	−868	202	**614** **595**
(4^3)	200	−144	24	80	−288	160	112	−28	−36	112	−42	−84
(51^7)	−917	524	−42	−374	1062	−534	−261	108	140	−272	52	104
(521^5)	5135	−3039	252	2052	−6020	3164	1500	−658	−776	1566	−312	−612
(52^21^3)	−7953	4970	−436	−3035	9355	−5276	−2346	1128	1155	−2480	528	992
(52^21)	2980	−2042	200	1036	−3406	2160	816	−456	−408	952	−224	−376
(531^4)	−3806	2195	−168	−1627	4740	−2413	−1259	532	630	−1246	232	518
(5321^3)	8290	−5088	400	3343	−10509	5854	2913	−1386	−1253	2734	−544	−1232
(532^21)	−1890	1306	−104	−671	2302	−1530	−594	360	254	−607	122	275
(53^21)	−1735	1042	−72	−742	2464	−1350	−836	388	258	−608	112	360
(541^5)	2771	−1630	136	1205	−3426	1736	860	−352	−524	1008	−192	−384
(5421)	−3780	2448	−224	−1494	4756	−2731	−1272	599	632	−1437	312	618
(543)	1020	−660	48	422	−1480	873	527	−253	−160	413	−76	−274
(51^3)	−855	495	−35	−381	1065	−546	−249	112	170	−302	50	100
(5^2)	510	−340	22	194	−630	394	147	−81	−85	187	−28	−71
(61^6)	817	−474	42	327	−942	479	236	−98	−120	242	−52	−94
(621^4)	−3778	2296	−220	−1473	4401	−2370	−1109	502	549	−1154	272	448
(62^21^2)	4172	−2750	308	1519	−4806	2870	1182	−630	−575	1298	−368	−484
(62^2)	−676	532	−96	−208	712	−496	−156	96	89	−236	102	76
(631^3)	2639	−1558	136	1115	−3317	1722	903	−390	−427	882	−192	−368
(6321)	−3402	2280	−224	−1342	4367	−2667	−1165	677	491	−1117	308	446
(63^2)	450	−294	24	180	−627	405	198	−135	−57	135	−36	−63
(641^4)	−1782	1087	−114	−766	2241	−1146	−593	233	338	−695	164	273
(642)	1080	−810	132	384	−1292	856	314	−184	−171	440	−180	−152
(651)	930	−561	48	408	−1193	635	298	−143	−183	349	−72	−118
(6^2)	−180	126	−18	−72	228	−144	−54	36	33	−72	27	18
(71^5)	−712	408	−32	−285	834	−419	−221	86	102	−212	40	94
(721^3)	2601	−1570	128	1008	−3099	1666	839	−358	−365	802	−160	−368
(72^21)	−1764	1180	−104	−620	2049	−1289	−525	295	229	−536	122	235
(731^2)	−1663	970	−70	−708	2203	−1122	−680	260	261	−584	104	312
(732)	942	−640	44	343	−1191	809	326	−223	−119	281	−56	−134
(741)	971	−589	48	419	−1313	663	415	−139	−181	415	−76	−226
(75)	−355	225	−12	−149	485	−284	−142	71	65	−142	18	71
(81^4)	607	−352	32	243	−708	356	186	−72	−88	184	−40	−80
(821^2)	−1640	1023	−106	−623	1959	−1065	−543	224	226	−528	132	249
(82^2)	508	−386	60	160	−556	376	138	−72	−65	176	−66	−76
(831)	885	−533	48	373	−1227	640	418	−158	−131	328	−76	−202
(84)	−380	264	−36	−152	528	−304	−184	68	66	−184	58	116
(91^3)	−499	288	−22	−203	580	−300	−142	64	76	−148	28	56
(921)	864	−549	44	320	−1031	612	268	−140	−115	262	−56	−110
(93)	−315	201	−12	−126	447	−279	−153	90	39	−99	18	63
$(10, 1^2)$	400	−234	22	167	−472	237	115	−46	−67	130	−28	−47
$(10, 2)$	−340	250	−32	−112	380	−264	−82	56	45	−112	38	36
$(11, 1)$	−290	164	−12	−127	372	−177	−105	36	52	−105	18	47
(12)	180	−120	12	72	−240	144	72	−36	−30	72	−18	−36

Table 4.12 (*continued*)

$w-12$ (iv)	h_6^3	$h_5h_7^2$	$h_5h_6h_7h_1^2$	$h_5h_6^2h_7h_1^2$	$h_5h_6^2h_7$	$h_5h_6h_7h_1^4$	$h_5h_6h_7h_1^6$	$h_5h_6h_7^3$	$h_5h_6^3h_7$	$h_5h_6h_7h_1^3$	$h_5h_6h_7h_1$	$h_5h_6h_7$	$h_6^3h_7^2$
(1^{12})	34650	3991680	1995840	997920	498960	665280	332640	166320	110880	166320	83160	27720	33264
(21^{10})	22050	2298240	1164240	589680	298620	398160	201600	102060	68880	103320	52290	17850	21672
(2^21^8)	14070	1310400	673680	346248	177912	236880	121688	62496	42728	63924	32816	11508	14084
(2^31^6)	9000	740880	387090	202212	105615	140130	73182	38211	26472	39366	20550	7428	9120
(2^41^4)	5769	415800	221040	117540	62520	82440	43860	23340	16380	24120	12840	4800	5880
(2^5)	2385	128520	70920	39240	21780	28080	15600	8700	6240	8910	4980	2010	2466
(31^9)	11130	1028160	529200	272160	139860	187488	96264	49392	33936	51408	26334	9240	11592
(321^7)	7140	575400	301560	157878	82572	110404	57687	30114	21000	31533	16485	5971	7518
(32^21^3)	4590	320040	170950	91236	48651	64672	34464	18350	12984	19350	10291	3863	4848
(32^31^5)	2955	177030	96450	52545	28623	37686	20529	11181	8019	11766	6408	2502	3108
(3^21^6)	1905	97440	54180	30168	16824	21848	12192	6816	4944	7112	3981	1622	1980
(3^221^4)	3660	243600	131040	70308	37632	50536	26990	14384	10276	15550	8258	3112	4016
(3^231^2)	2364	133840	73530	40320	22067	29334	16036	8749	6346	9424	5132	2018	2568
(3^23^2)	1527	73220	41100	23062	12934	16940	9502	5326	3914	5676	3182	1310	1632
(3^31^3)	987	39900	22890	13158	7581	9734	5614	3247	2408	3398	1969	851	1030
(3^321)	1230	54600	30990	17544	9903	13092	7386	4152	3096	4524	2541	1059	1344
(3^4)	795	29610	17190	9981	5793	7494	4356	2529	1907	2697	1569	689	846
(41^8)	415	11760	7080	4272	2580	3264	1980	1200	928	1260	768	364	432
	4830	393120	205800	107604	56196	75768	39508	20580	14420	22134	11494	4158	5446
(421^6)	3120	215040	115095	61506	32820	44059	23470	12486	8898	13534	7177	2696	3526
(42^21^4)	2019	117180	64150	35079	19161	25492	13907	7579	5487	8214	4466	1750	2264
(42^31^2)	1308	63630	35640	19965	11187	14676	8219	4605	3379	4950	2771	1137	1442
(42^4)	849	34440	19740	11340	6534	8408	4844	2802	2076	2963	1715	739	911
(431^5)	1620	86040	48185	26598	14628	19720	10806	5903	4332	6603	3586	1414	1888
(4321^3)	1053	47005	26675	15099	8525	11311	6374	3583	2670	3958	2218	920	1196
(432^21)	684	25340	14725	8556	4972	6456	3750	2179	1642	2357	1369	599	752
(43^21^2)	555	18550	10940	6435	3773	4942	2897	1691	1299	1868	1090	486	618
(43^22)	360	9940	6005	3633	2201	2797	1697	1031	797	1100	670	317	384
(4^31^3)	729	29190	16870	9684	5526	7474	4238	2392	1814	2780	1554	648	892
(4^221^2)	477	15610	9250	5460	3212	4238	2484	1452	1116	1642	955	423	556
(4^23^2)	312	8330	5060	3074	1870	2392	1452	884	684	964	586	276	344
(4^31)	255	6020	3720	2294	1411	1814	1116	684	542	756	464	225	280
(4^2)	120	1890	1230	801	522	648	423	276	225	297	195	105	123
	20												
(51^7)		130872	70056	37422	19950	27076	14399	7644	5488	8533	4508	1694	2324
	−8	74											
(521^5)			38491	21102	11544	15541	8483	4621	3380	5172	2805	1103	1498
	48	−393	2149										
(52^21^3)				11886	6687	8874	4986	2798	2080	3108	1739	719	956
	−80	567	−3218	5090									
(52^21)					3879	5042	2923	1698	1277	1853	1075	469	604
	32	−191	1142	−1974	920								
(531^4)						6794	3845	2166	1640	2493	1395	583	800
	−40	289	−1598	2375	−801	1290							
(5321^2)							2251	1314	1009	1476	859	381	502
	96	−580	3351	−5329	1970	−2676	6162						
(532^2)								800	618	868	528	249	312
	−20	119	−739	1340	−666	545	−1444	566					
(53^21)									490	683	419	203	256
	−28	115	−678	1085	−358	590	−1447	278	453				
(541^3)										1035	602	270	378
	32	−204	1147	−1727	615	−958	1919	−409	−383	826			
(5421)											368	177	232
	−56	240	−1426	2360	−971	1150	−2664	695	572	−960	1422		
(543)												95	114
	24	−58	358	−620	255	−310	814	−215	−253	227	−400	202	
(5^21^2)													160
	−6	68	−391	600	−215	340	−694	159	133	−306	336	−71	136
(52^2)	4	−34	214	−390	198	−170	434	−176	−71	153	−247	59	−68
(61^6)	8	−66	346	−492	161	−249	495	−94	−100	169	−200	48	−53
(621^4)	−40	286	−1551	2295	−786	1135	−2361	470	490	−788	980	−240	255
(62^21^2)	48	−289	1637	−2572	954	−1187	2658	−578	−560	839	−1141	281	−280
(62^3)	−12	38	−228	396	−184	150	−360	100	64	−120	192	−40	35
(631^3)	32	−196	1079	−1585	513	−875	1797	−327	−409	635	−762	199	−216
(6321)	−40	238	−1382	2191	−755	1106	−2573	505	632	−760	1046	−292	277
(63^2)	4	−30	180	−291	87	−159	405	−63	−135	93	−135	51	−36
(641^2)	−28	124	−698	1043	−358	592	−1171	214	248	−516	603	−139	182
(642)	24	−60	360	−598	236	−286	656	−128	−140	244	−368	80	−77
(651)	8	−70	408	−629	211	−366	763	−142	−169	331	−373	83	−148
(65)	−2	12	−72	114	−36	66	−144	18	36	−60	72	−12	27
(71^5)	−8	58	−304	438	−151	213	−435	94	88	−139	176	−48	41
(721^3)	32	−195	1067	−1623	605	−758	1646	−397	−340	508	−694	199	−155
(72^21)	−20	119	−692	1153	−501	476	−1145	351	224	−335	514	−150	110
(731^2)	−28	119	−665	1011	−360	534	−1162	262	275	−369	504	−166	116
(732)	8	−62	377	−657	295	−286	741	−251	−161	195	−316	110	−76
(741)	24	−62	359	−568	223	−302	653	−158	−150	259	−363	115	−82
(75)	−4	24	−149	260	−118	130	−319	106	71	−118	177	−59	53
(81^4)	8	−50	262	−375	123	−185	379	−73	−81	118	−148	41	−34
(821^2)	−28	120	−663	1017	−365	471	−1057	227	240	−300	429	−131	86
(82^2)	12	−30	180	−318	156	−114	288	−100	−52	84	−144	40	−23
(831)	24	−60	341	−524	166	−281	657	−115	−193	170	−257	107	−47
(84)	−20	20	−120	200	−80	100	−240	52	68	−80	136	−56	18
(91^3)	−4	42	−222	319	−107	161	−323	65	65	−110	132	−33	34
(921)	8	−63	358	−569	212	−257	605	−139	−137	167	−251	75	−52
(93)	−4	21	−126	210	−75	105	−279	63	90	−57	99	−51	18
$(10, 1^2)$	4	−33	177	−256	89	−134	260	−56	−47	101	−114	24	−34
$(10, 2)$	−4	22	−132	230	−108	90	−224	76	36	−68	112	−24	23
$(11, 1)$	−4	23	−127	186	−59	104	−210	36	47	−81	94	−24	29
(12)	4	−12	72	−120	48	−60	144	−36	−36	48	−72	24	−18

Table 4.12 (*continued*)

$w=12$ (v)	$h_0^2h_2$	$h_0h_1{}^6$	$h_0h_2h_1{}^4$	$h_0h_2{}^2h_1{}^2$	$h_0h_2{}^3$	$h_0h_1h_1{}^3$	$h_0h_2h_1{}^2h_1$	$h_0h_3{}^2$	$h_0h_1{}^2h_1{}^2$	$h_0h_2h_1$	$h_0h_1{}^4$	$h_1{}^2h_1{}^2$	$h_1h_2h_1{}^2$
(1^{12})	16632	665280	332640	166320	83160	110880	55440	18480	27720	13860	5544	924	95040	47520
(21^{10})	10962	408240	206640	104580	52020	70560	35700	12180	18270	9240	3822	672	62640	31680
(2^21^8)	7224	246960	126840	65128	33432	44520	22848	8008	11984	6146	2632	490	40560	20808
(2^31^6)	4758	147510	77040	40227	21000	27870	14547	5257	7821	4080	1809	358	25830	13482
(2^41^4)	3132	87120	46356	24672	13134	17316	9220	3448	5076	2704	1240	262	16200	8628
(2^51^2)	2061	50940	27660	15040	8190	10680	5820	2260	3275	1790	847	192	10020	5460
(2^6)	1356	29520	16380	9120	5100	6540	3660	1480	2100	1185	576	141	6120	3420
(31^9)	5922	206640	103840	54180	27720	37128	18984	6636	10038	5124	2226	420	36000	18360
(321^7)	3906	121800	63560	33138	17262	23079	12012	4340	6538	3395	1533	308	22620	11753
(32^21^5)	2572	71070	37860	20151	10716	14252	7574	2839	4235	2246	1052	226	14000	7436
(32^31^3)	1692	41100	22386	12192	6639	8741	4760	1858	2726	1484	719	166	8550	4656
(32^41)	1113	23580	13148	7344	4110	5324	2082	1216	1743	980	489	122	5160	2888
(3^31^6)	2120	57200	30560	16288	8662	11646	6180	2326	3530	1863	896	196	11960	6338
(3^321^4)	1392	32750	17928	9793	5338	7112	3871	1521	2268	1229	612	144	7210	3928
$(3^32^21^2)$	914	18620	10454	5866	3289	4312	2418	996	1446	810	416	106	4300	2414
(3^32^3)	601	10520	6062	3502	2029	2596	1506	652	915	534	281	78	2540	1472
(3^321^3)	750	14550	8244	4653	2616	3480	1953	813	1200	669	354	92	3540	1998
(3^321)	492	8160	4752	2766	1608	2085	1214	534	756	440	239	68	2070	1209
(3^3)	264	3480	2112	1284	780	988	604	288	388	238	136	44	960	588
(41^8)	2814	93240	48300	25004	12936	17388	8988	3220	4914	2534	1162	238	19080	9804
(421^6)	1860	53265	28250	14966	7920	10614	5610	2093	3174	1672	801	176	11565	6086
(42^21^4)	1226	30210	16431	8929	4848	6443	3495	1363	2037	1102	549	130	6920	3743
(42^31^2)	807	17025	9507	5312	2970	3887	2173	889	1298	726	374	96	4095	2283
(42^4)	531	9540	5474	3152	1824	2330	1348	580	821	479	253	71	2400	1382
(431^5)	1016	23525	12900	7048	3838	5180	2811	1107	1693	911	471	114	5720	3098
(4321^3)	666	13160	7424	4177	2344	3116	1745	723	1076	599	320	84	3345	1874
(432^21)	437	7325	4253	2470	1435	1861	1081	473	678	394	216	62	1940	1126
(43^21^2)	358	5580	3286	1927	1125	1484	865	385	560	324	184	54	1550	910
(43^22)	235	3075	1866	1134	690	876	534	253	348	213	123	40	885	540
(4^21^4)	492	8930	5092	2886	1627	2206	1236	518	800	442	250	68	2510	1406
(4^221^2)	321	4910	2890	1695	991	1310	763	339	502	290	168	50	1430	834
(4^23^2)	210	2690	1634	994	607	772	470	222	312	191	112	37	810	492
(4^231)	171	2010	1244	767	471	610	374	181	256	156	95	32	630	390
(4^2)	81	690	453	297	195	243	159	85	114	75	48	19	240	159
(51^7)	1218	37632	19796	10402	5460	7371	3864	1428	2198	1148	560	126	9276	4817
(521^5)	806	20841	11286	6102	3294	4421	2383	925	1407	755	386	94	5411	2897
(52^21^3)	532	11490	6411	3575	1992	2636	1468	601	893	496	264	70	3126	1711
(52^31)	351	6309	3629	2092	1209	1562	903	391	562	326	179	52	1791	1028
(531^4)	442	8645	4896	2758	1546	2089	1166	485	741	409	228	62	2486	1387
(5321^2)	290	4724	2762	1611	937	1234	717	317	464	268	154	46	1411	819
(532^2)	191	2573	1553	940	571	724	440	207	288	176	103	34	796	481
(53^21)	156	1908	1174	721	441	569	349	169	236	144	88	30	616	379
(541^3)	216	3014	1802	1068	628	850	496	224	344	197	122	38	991	581
(5421)	141	1628	1007	621	382	496	304	147	212	129	81	28	551	338
(543)	75	642	420	275	180	224	147	79	105	69	45	18	231	152
(5^21^2)	96	926	590	372	232	310	192	96	144	87	60	22	352	220
(5^22)	53	494	326	215	142	178	117	63	87	57	39	16	192	126
(61^6)	77	13327	7186	3865	2074	2820	1509	583	909	483	255	64	4051	2140
(621^4)	24	64	4011	2233	1240	1666	923	378	576	317	175	48	2276	1250
(62^21^2)	−120	−276	1246	1290	745	977	564	246	361	208	119	36	1271	727
(62^3)	142	279	−1338	1584	451	569	344	160	224	137	80	27	706	421
(631^3)	−22	−38	198	−276	84	766	444	198	299	171	103	32	971	565
(6321)	93	186	−858	909	−120	675	271	130	184	112	69	24	536	326
(63^2)	−127	−228	1137	−1381	200	−875	1460	70	91	60	39	16	221	144
(641^2)	12	30	−159	207	−24	138	−279	84	136	82	55	20	351	219
(642)	−76	−114	543	−593	98	−451	545	−72	383	54	36	15	191	125
(651)	34	60	−318	430	−108	244	−408	60	−210	220	27	12	112	76
(6^2)	59	60	−301	350	−48	266	−375	60	−231	120	196	7	32	24
(71^5)	−6	−12	66	−90	18	−60	108	−24	54	−48	−48	21	1546	841
(721^3)	−24	−57	240	−234	30	−156	180	−21	90	−42	−42	6	57	480
(72^21)	103	190	−842	873	−120	561	−695	84	−335	168	164	−24	−190	680
(731^2)	−91	−114	543	−633	100	−348	510	−63	215	−128	−118	18	114	−450
(732)	−76	−114	516	−518	62	−406	473	−63	257	−114	−120	18	114	−423
(741)	69	57	−285	345	−44	208	−342	51	−117	68	83	−12	−57	242
(75)	59	57	−267	274	−40	224	−233	24	−189	80	94	−12	−57	224
(81^4)	−47	−19	95	−107	12	−83	107	−12	71	−24	−59	6	10	−83
(821^2)	17	50	−212	213	−30	135	−166	21	−76	42	35	−6	−50	162
(82^2)	−51	−120	543	−594	98	−351	478	−63	204	−134	−94	18	120	−423
(831)	22	30	−150	198	−56	84	−128	12	−62	60	24	−6	−30	120
(84)	24	60	−281	303	−40	221	−303	51	−126	80	59	−12	−60	221
(91^3)	−12	−20	100	−120	28	−80	104	−12	68	−56	−24	6	20	−80
(921)	−17	−42	180	−181	22	−119	150	−21	68	−34	−35	6	42	−138
(93)	34	63	−295	337	−44	194	−305	51	−104	68	59	−12	−63	232
$(10, 1^2)$	−12	−21	105	−126	12	−84	153	−39	36	−24	−24	6	21	−84
$(10, 2)$	17	33	−144	145	−22	101	−114	12	−68	34	35	−6	−33	111
$(11, 1)$	−22	−22	110	−142	32	−68	112	−12	46	−44	−24	6	22	−88
(12)	−12	−23	104	−105	12	−81	94	−12	58	−24	−35	6	23	−81
	12	12	−60	72	−12	48	−72	12	−36	24	24	−6	−12	48

Table 4.12 (*continued*)

$w-12$ (vi)	$h_7h_5^2h_1$	$h_7h_5h_3h_1^2$	$h_7h_5h_3$	$h_7h_4h_1$	h_7h_3	$h_6h_1^4$	$h_6h_5h_1^3$	$h_6h_3^2h_1^2$	$h_6h_5h_3$	$h_6h_4h_1$	h_6h_4	$h_6h_2^2$	$h_6h_3h_1$	h_6h_2	$h_{10}h_1^2$	$h_{10}h_2$	$h_{11}h_1$	h_{12}
(1^{12})	23760	15840	7920	3960	792	11880	5940	2970	1980	495	1320	660	220	132	66	12	1
(21^{10})	16020	10800	5460	2790	582	8460	4275	2160	1455	375	1020	515	175	111	56	11	1
(2^21^8)	10672	7288	3736	1956	428	5910	3028	1551	1058	283	774	396	138	92	47	10	1
(2^31^6)	7035	4872	2541	1365	315	4050	2112	1101	762	213	576	300	108	75	39	9	1
(2^41^4)	4596	3228	1720	948	232	2724	1452	774	544	160	420	224	84	60	32	8	1
(2^5)	2980	2120	1160	655	171	1800	985	540	385	120	300	165	65	47	26	7	1
(31^9)	1920	1380	780	450	126	1170	660	375	270	90	210	120	50	36	21	6	1
(321^7)	9360	6384	3252	1704	372	5508	2799	1422	966	255	748	379	130	91	46	10	1
(32^21^4)	6102	4232	2194	1185	274	3734	1932	999	690	191	553	285	101	74	38	9	1
	3946	2784	1475	821	202	2482	1314	695	489	143	400	211	78	59	31	8	1
(32^21^3)	2535	1817	989	566	149	1620	882	480	344	107	283	154	60	46	25	7	1
(32^41)	1620	1176	662	388	110	1040	585	330	240	80	196	111	46	35	20	6	1
(3^21^6)	3352	2380	1254	708	176	2246	1180	619	436	127	380	198	72	58	30	8	1
(3^221^5)	2135	1546	837	488	130	1446	783	423	305	95	266	143	55	45	24	7	1
(3^231^3)	1354	996	558	334	96	916	514	288	212	71	182	102	42	34	19	6	1
(3^241)	856	636	372	227	71	572	334	196	146	53	122	72	32	25	15	5	1
(3^31^3)	1122	840	468	288	84	798	447	249	186	63	168	93	38	33	18	6	1
(3^321)	705	534	311	195	62	492	288	168	128	47	111	65	29	24	14	5	1
(3^3)	360	280	172	112	40	252	156	96	76	31	64	40	20	16	10	4	1
(41^8)	5036	3468	1780	954	218	3393	1733	885	605	183	529	269	93	73	37	9	1
(421^6)	3200	2250	1181	657	161	2219	1160	606	423	121	379	197	71	58	30	8	1
(42^21^4)	2023	1451	783	451	119	1423	766	412	294	90	265	142	54	45	24	7	1
(42^31^2)	1274	920	519	308	88	897	500	279	203	67	181	101	41	34	19	6	1
(42^4)	800	590	344	209	65	557	323	189	139	50	121	71	31	25	15	5	1
(431^5)	1673	1216	653	386	104	1255	671	358	257	79	248	131	49	44	23	7	1
(4321^3)	1047	777	432	264	77	779	433	240	177	59	167	92	37	33	18	6	1
(432^21)	654	492	286	170	57	477	277	161	121	44	110	64	28	24	14	5	1
(43^21^2)	531	410	237	154	50	403	235	136	105	39	99	57	25	23	13	5	1
(43^22)	330	256	157	103	37	241	148	91	71	29	63	39	19	16	10	4	1
(4^31^3)	784	590	326	206	62	648	356	195	144	48	152	82	32	32	17	6	1
(4^221^2)	485	372	215	140	46	388	224	129	98	36	98	56	24	23	13	5	1
(4^231)	300	232	142	94	34	230	140	86	66	27	62	38	18	16	10	4	1
(4^32)	240	192	117	81	30	188	116	71	57	24	54	33	16	15	9	4	1
(4^4)	105	87	57	42	18	81	54	36	30	15	27	18	10	9	6	3	1
(51^7)	2500	1744	904	498	120	1961	1009	519	358	99	358	183	64	57	29	8	1
(521^5)	1549	1108	591	340	89	1231	653	346	245	73	247	130	48	44	23	7	1
(52^21^3)	958	700	387	231	66	759	418	230	167	54	166	91	36	33	18	6	1
(52^31)	592	439	254	156	49	461	265	153	113	40	109	63	27	24	14	5	1
(531^4)	770	576	317	197	58	647	355	194	143	47	152	82	32	32	17	6	1
(5321^2)	474	361	208	133	43	387	223	128	97	35	98	56	24	23	13	5	1
(532^2)	292	224	137	89	32	229	139	85	65	26	62	38	18	16	10	4	1
(53^21)	232	184	112	76	28	187	115	70	56	23	54	33	16	15	9	4	1
(541^3)	338	265	152	103	35	304	174	99	76	28	86	48	20	22	12	5	1
(5421)	207	164	100	69	26	176	107	65	51	21	53	32	15	15	9	4	1
(543)	100	82	54	39	17	80	53	35	29	14	27	18	10	9	6	3	1
(5^21^3)	136	112	68	50	20	128	78	47	38	16	44	26	12	14	8	4	1
(5^22)	83	68	45	33	15	72	47	31	25	12	26	17	9	9	6	3	1
(61^6)	1120	804	423	243	63	1045	544	283	198	57	229	118	42	43	22	7	1
(621^4)	685	502	274	165	47	627	340	184	133	42	151	81	31	32	17	6	1
(62^21^2)	416	311	178	111	35	371	211	120	89	31	97	55	23	23	13	5	1
(62^3)	253	191	116	74	26	217	130	79	59	23	61	37	17	16	10	4	1
(631^3)	326	253	144	95	31	303	173	98	75	27	86	48	20	22	12	5	1
(6321)	198	155	94	63	23	175	106	64	50	20	53	32	15	15	9	4	1
(63^2)	94	76	50	35	15	79	52	34	28	13	27	18	10	9	6	3	1
(641^2)	135	111	67	49	19	128	78	47	38	16	44	26	12	14	8	4	1
(642)	82	67	44	32	14	72	47	31	25	12	26	17	9	9	6	3	1
(651)	51	44	30	23	11	48	32	21	18	9	20	13	7	8	5	3	1
(65)	18	16	12	10	6	16	12	9	8	5	8	6	4	4	3	2	1
(71^5)	456	336	181	111	32	501	266	141	101	31	136	71	26	31	16	6	1
(721^3)	273	207	117	75	24	287	161	90	67	23	85	47.	19	22	12	5	1
(72^21)	**164** / **366**	126	76	50	18	163	97	58	44	17	52	31	14	15	9	4	1
(731^2)	263	**102** / **329**	61	43	16	127	77	46	37	15	44	26	12	14	8	4	1
(732)	−216	−165	**40** / **172**	28	12	71	46	30	24	11	26	17	9	9	6	3	1
(741)	−148	−189	83	**22** / **154**	10	48	32	21	18	9	20	13	7	8	5	3	1
(75)	71	71	−59	−59	**6** / **47**	16	12	9	8	5	8	6	4	4	3	2	1
(81^4)	−93	−93	43	43	−12	**209** / **50**	115	63	47	16	73	39	15	21	11	5	1
(821^2)	267	255	−129	−126	36	−120	**68** / **327**	40	31	12	43	25	11	14	8	4	1
(82^2)	−100	−62	44	40	−12	30	−98	**26** / **56**	20	9	25	16	8	9	6	3	1
(831)	−126	−177	75	91	−24	60	−177	40	**17** / **143**	8	20	13	7	8	5	3	1
(84)	52	68	−24	−56	12	−20	68	−28	−56	**5** / **44**	8	6	4	4	3	2	1
(91^3)	85	77	−43	−35	12	−42	96	−22	−44	12	**34** / **42**	19	8	13	7	4	1
(921)	−168	−131	95	59	−24	63	−169	44	86	−24	−63	**12** / **124**	6	8	5	3	1
(93)	63	63	−51	−24	12	−21	63	−12	−51	12	21	−51	**4** / **39**	4	3	2	1
$(10, 1^2)$	−67	−68	34	35	−12	33	−78	22	35	−12	−33	45	−12	**7** / **33**	4	3	1
$(10, 2)$	76	46	−44	−24	12	−22	66	−32	−24	12	22	−44	12	−22	**3** / **32**	2	1
$(11, 1)$	47	58	−24	−35	12	−23	58	−12	−35	12	23	−35	12	−23	12	**2** / **23**	1
(12)	−36	−36	24	24	−12	12	−36	12	24	−12	−12	24	−12	12	−12	−12	**1** / **12**

MISCELLANEA

Some procedures for comparing Poisson processes or populations

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1. Introduction

The problem of comparing two Poisson processes arises in a variety of applications. For example, if a manufactured substance like cloth, paper, or wire is inspected continuously for flaws, the number x of faults observed in any length t may often be assumed to have the probability distribution

$$p(x, \lambda, t) = e^{-\lambda t}(\lambda t)^x/x! \quad (x = 0, 1, 2, \ldots),$$

where λ is the mean number of faults per unit length. Hence the problem of comparing two kinds of material as to flaws is statistically a problem of comparing the parameters λ_1, λ_2 of two Poisson processes.

Alternatively, if a Geiger counter is used to record emissions from a radioactive substance under essentially constant conditions, the number x of emissions recorded during any duration of time t may be assumed to have the probability distribution $p(x, \lambda, t)$, where λ is the mean number of emissions recorded per unit interval of time. The problem of comparing the emission rates of two such substances observed under comparable conditions is statistically again the problem of comparing the parameters λ_1, λ_2 of two Poisson processes.

2. Some experimental designs

A simple experimental procedure which will be convenient for many such problems is the simultaneous observation of the two processes. Assume that two such processes characterized by distributions $p(x_1, \lambda_1, t)$, $p(x_2, \lambda_2, t)$ are observed simultaneously until the first 'event' occurs, i.e. until either $x_1 = 1$ or $x_2 = 1$; in terms of the above examples, an 'event' consists of the observation of a flaw or the recording of a count. (The simultaneous occurrence of two or more 'events', either in the same or in different processes, has probability zero; hence this possibility may be ignored.) Then the probability that the first event occurs in the first process, on the condition that between t and $t + \Delta t$ units of inspection are performed before it occurs, is

$$\lambda_1/(\lambda_1 + \lambda_2 + \epsilon),$$

where ϵ approaches 0 with Δt; this is seen to be independent of t. Hence if p is the probability that the first event is observed in the first process, $p = \lambda_1/(\lambda_1 + \lambda_2) = [(1 + (\lambda_2/\lambda_1)]^{-1}$, which depends on λ_1, λ_2 only through their ratio $\gamma = \lambda_2/\lambda_1$. If such simultaneous inspection is continued, and if we set $y_i = 1$ or 0 according as the ith event is observed in the first or second process, the experiment is seen to provide a sequence of independent Bernoulli observations y_i such that

$$\Pr\{y_i = 1\} = p,$$
$$\Pr\{y_i = 0\} = 1 - p \quad (i = 1, 2, \ldots),$$

where $p = 1/(1 + \gamma)$.

For many (but not all) purposes it is appropriate to express comparisons of the two processes in terms of the ratio $\gamma = \lambda_2/\lambda_1$. Statistical questions concerning γ may then be answered by use of the various methods available for dealing with Bernoulli observations.

Problem 1. Suppose that it is required to test whether $\lambda_1 = \lambda_2$ against the alternatives $\lambda_1 \geqslant \lambda_2$, with significance level $\alpha = 0{\cdot}05$, and with power $1 - \beta = 0{\cdot}95$ when $\lambda_2 = \frac{1}{2}\lambda_1$. The requirement may be met by any test, based on the observations y_i, of the hypothesis that $p = \frac{1}{2}$ against the alternatives $p > \frac{1}{2}$, with significance level $\alpha = 0{\cdot}05$, and power $1 - \beta = 0{\cdot}95$ when $p = \frac{2}{3}$.

Method 1. Using tables of the binomial distribution (or tables of the normal distribution for approximation) we find the smallest value of n such that, for some corresponding critical value r_n, we have

$$\Pr\left\{\sum_{i=1}^{n} y_i > r_n \,|\, p = \tfrac{1}{2}\right\} \leqslant 0{\cdot}05$$

and

$$\Pr\left\{\sum_{i=1}^{n} y_i > r_n \,|\, p = \tfrac{2}{3}\right\} \geqslant 0{\cdot}95.$$

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(By use of the randomization procedure described, for example, in Tocher (1950), these inequalities can be replaced by equalities, and these requirements may be met by a smaller value of n than before.) Using the normal approximation, we find these requirements satisfied approximately by $n = 130$ and $r_n = 77$.

The appropriate procedure is, therefore, to observe the two processes simultaneously until a total of 130 events has occurred, and then to reject the hypothesis that $\lambda_1 = \lambda_2$ if and only if 78 or more of the events occurred in the first process. An appreciable shortening of the average duration of the experiment can be achieved by the following modification:

Method 2. Without altering the size or power of the test we modify the preceding method to obtain the following curtailed-sampling procedure: Observe the processes until either (*a*) the number of events observed in the first process equals 78, in which case reject the hypothesis, or (*b*) the number of events observed in the second process equals 53, in which case accept the hypothesis.

Method 3. The average duration of the experiment can be minimized, whether $\lambda_1 = \lambda_2$ or $\lambda_2 = \frac{1}{2}\lambda_1$, by use of Wald's (1947) sequential probability ratio test. Let

$$a = \log\frac{1-\beta}{\alpha} = \log\frac{0{\cdot}95}{0{\cdot}05} = \log 19, \quad b = \log\frac{\beta}{1-\alpha} = \log\frac{0{\cdot}05}{0{\cdot}95} = -a,$$

$$h_1 = \frac{a}{\log\dfrac{p_1(1-p_0)}{p_0(1-p_1)}} = \frac{a}{\log 2}, \quad h_0 = \frac{b}{\log\dfrac{p_1(1-p_0)}{p_0(1-p_1)}} = -h_1,$$

and

$$s = \log\frac{1-p_0}{1-p_1} = \log\tfrac{2}{1}.$$

For every $m = 1, 2, \ldots$, let z_m be the number of events observed in the first process, out of the first m events observed; i.e. $z_m = \sum_{i=1}^{m} y_i$. Continue the experiment until either

(*a*) $z_m \geqslant h_1 + ms$, in which case reject the hypothesis $\lambda_1 = \lambda_2$, or

(*b*) $z_m \leqslant h_0 + ms$, in which case accept the hypothesis.

Problem 2. Suppose that it is desired to give an estimate of $\gamma = \lambda_2/\lambda_1$ by means of a confidence interval. We may obtain n observations $y_1, \ldots, y_n$ and use the procedure of Clopper & Pearson (1934) to determine a confidence interval (p_L, p_U) for p such that $\Pr\{p_L < p < p_U\} \geqslant 1-\beta$. Then since $\gamma = 1/p - 1$ we have $\Pr\{1/p_U - 1 < \lambda_2/\lambda_1 < 1/p_L - 1\} \geqslant 1-\beta$, so that $(1/p_U - 1,\ 1/p_L - 1)$ is the desired confidence interval.

A useful property of all procedures based on the y_i's is that even if λ_1 and λ_2 vary during the course of an experiment, so long as λ_2/λ_1 remains constant the procedures remain valid.

The generalization of these procedures to problems of comparing more than two processes is direct; binomial probability distributions of the y_i's will then be replaced by multinomial distributions.

3. Duration of the experiment

In the procedure described above to obtain the sequence of observations y_i, let t_1 denote the amount of inspection performed to obtain the first observation, and generally let t_i denote the amount of inspection performed after y_{i-1} is obtained to obtain y_i, for $i = 2, 3, \ldots$. If we denote by m the number of observations y_i obtained during any amount t of inspection, then the probability distribution of m is given by

$$p(m, \mu, t) = e^{-\mu t}\frac{(\mu t)^m}{m!} \quad (m = 0, 1, 2, \ldots),$$

where $\mu = \lambda_1 + \lambda_2$. The t_i's are observed 'waiting-times' between events in this Poisson process. As is shown, for example, in Kendall (1950), the t_i's are distributed according to the density function

$$f(t, \mu) = \mu e^{-\mu t} \quad (t \geqslant 0).$$

Thus $2\mu t$ is distributed as χ^2_2 (χ^2 with 2 D.F.); letting

$$T_n = \sum_{i=1}^{n} t_i$$

the amount of inspection required for a procedure based on n observations y_i, we have that $2\mu T_n$ is distributed as χ^2_{2n}. If, for example, $n = 10$, we find (on referring to tables of the distribution of χ^2_{20}) that $\Pr\{T_n < 31{\cdot}4/(2\mu)\} = 0{\cdot}95$. If $\mu = \frac{1}{2}$, then with probability 0·95, less than 31·4 units of inspection will suffice. Such calculations are of use in procedures like those proposed above for Problem 2, and in

Method 1 for Problem 1. For Method 2 under Problem 1, the same calculations may be used to provide a lower bound for the probability that a specified amount of inspection will suffice; thus in the preceding example, the probability that less than 31·4 units of inspection will suffice is appreciably more than 0·95 if Method 2 is used.

If Method 3 is used to minimize the average amount of inspection required for Problem 1, the formulae given by Wald (1947) may be used to obtain $E(n)$, the average number of observations y_i required. Since $1/\mu$ is the mean amount of inspection for one observation y_i, the average amount of inspection will be $E(n)/\mu$.

In all of these procedures it is clear that the average amount of inspection required varies inversely with $\mu = \lambda_1 + \lambda_2$, the 'nuisance-parameter' in our problem. If no lower bound on μ is known at the outset of inspection, no probability statements concerning the required amount of inspection can be made. At any time after inspection has begun, m/t constitutes an estimate of μ and so may be used to obtain rough estimates of the additional amount of inspection required.

4. Comparing Poisson populations

The problem of comparing two Poisson populations occurs in a number of applications in which it is somewhat unnatural to conceive of the observations as if obtained by inspection of a continuing process. For example, the numbers x_1, x_2 of cases of a rare disease observed in two large groups of individuals during some period may be assumed to be independently distributed according to the Poisson laws

$$g(x_i, \lambda_i) = e^{-\lambda_i}(\lambda_i)^{x_i}/x_i! \quad (x_i = 0, 1, 2, \ldots, \text{for } i = 1 \text{ and } 2).$$

Problems of comparing incidence rates in two such groups may *for some purposes* be formulated as statistical questions concerning the ratio $\gamma = \lambda_2/\lambda_1$. As

$$g(x_i, \lambda_i) = p(x_i, \lambda_i, 1) \quad (i = 1, 2),$$

such statistical problems are formally equivalent to problems involving Poisson processes where certain restrictions are imposed on the units and the amount of inspection which will be performed.

Each of the statistical methods proposed above for Poisson processes can be directly adapted to the present case. To apply Method 1 for Problem 1, for example, we should continue taking *paired* observations (x_1, x_2) until the total $\Sigma x_1 + \Sigma x_2$ of all observations reaches or exceeds the predetermined $n = 130$. Then the appropriate test based on binomial probabilities may be used.

Other applications have been discussed recently by Maguire, Pearson & Wynn (1952).

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Scale factors and degrees of freedom for small sample sizes for χ-approximation to the range

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Patnaik (1950) has approximated the distribution function of the range in random samples from a normal population by the use of the χ-distribution. If $\overline{w}_{m,n}$ is the average of the ranges of m independent samples of n then

$$\overline{w}_{m,n}/\sigma = c\chi/\sqrt{\nu}, \tag{1}$$

where c is a scale factor and ν is an equivalent number of degrees of freedom for χ. It follows that $\overline{w}_{m,n}/c$ may be considered to be equivalent to the usual standard deviation estimator s with ν degrees of freedom.

The series approximation which was used to obtain Patnaik's table of c and ν for various sample sizes leads to considerable error for small values of ν*. A corrected table of c and ν was given by H. A. David (1951, Table I) but only to the nearest 0·01 in c and 0·1 in ν. This note presents in Table 1 new values of c and ν for the single sample case ($m = 1$) based on recently published moment constants for the distribution of the range in normal samples (Hartley & Pearson, 1951) and an interpolation of the Γ-function at close intervals.

It was noticed that if these corrected values of c and ν are employed, the approximation to the distribution of range by $c\chi/\sqrt{\nu}$ is somewhat better than was indicated by Patnaik (1950). Thus results of Lord (1947), obtained exactly by quadrature, provide factors by which the range (w) in a single sample of n should be multiplied to obtain confidence limits for the mean (μ) of the normal population from which the sample has been drawn. For example, 95 % confidence limits can be obtained from

$$\bar{x} \pm f_{0\cdot05} w,$$

where $\bar{x}$ is the sample mean and $f_{0\cdot05}$ is given in the last column of Table 1 below. If $t_{0\cdot05}$ is the 5 % limit of Student's ratio (two-sided test) for ν degrees of freedom, then $f_{0\cdot05}$ may be approximated by $t_{0\cdot05}/(c\sqrt{n})$. Comparison of the last two columns of the table shows that the agreement is excellent. The differences are negligible for all practical purposes. The agreement is not unexpected in view of Pearson's (1952) findings that the Patnaik χ method provides an excellent approximation to the probability integral of the range for $n = 4, 6, 10, 15$.

Table 1. *Scale factors and equivalent degrees of freedom for χ-approximation to range in normal samples*

No. in sample n	Degrees of freedom ν	Scale factor c	Equivalent two-sided 5 % t	Confidence limits*	
				$\pm t/(c\sqrt{n})$	Lord†
2	1·0000	$1{\cdot}41421_4$	12·7062	6·3531	6·3531
3	$1{\cdot}9845_9$	$1{\cdot}91154_7$	4·3349	1·3093	1·3039
4	$2{\cdot}9291_2$	$2{\cdot}23886_6$	3·2265	0·7206	0·7166
5	$3{\cdot}8266_{5+}$	$2{\cdot}48124_2$	2·8267	0·5095	0·5066
6	$4{\cdot}6772_3$	$2{\cdot}67252_6$	2·6249	0·4010	0·399
7	$5{\cdot}4841_0$	$2{\cdot}82980_8$	2·5038	0·3344	0·333
8	$6{\cdot}2511_7$	$2{\cdot}96288_3$	2·4233	0·2892	0·288
9	$6{\cdot}9818_4$	$3{\cdot}07793_7$	2·3658	0·2562	0·255
10	$7{\cdot}6798_{5-}$	$3{\cdot}17905_3$	2·3228	0·2311	0·2301

* Multipliers for sample range to get 95 % confidence limits of the mean.
† From direct quadrature (Lord, 1947).

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* Compare the values of c and ν given below with those in Patnaik's Table 1 for $m = 1$.

The third moment of Gini's mean difference

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U. S. Nair (1936) has given an expression for the variance of Gini's mean difference, defined by

$$g = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} |x_i - x_j|. \tag{1}$$

Nair's formula applies to any parental distribution of the x_i, although its actual evaluation may be tedious. Recently Lomnicki (1952) has obtained a simpler expression for the same by a more direct approach. In this note we derive the third moment of g when the parental distribution is normal with the help of the absolute moments of normal distribution obtained by Nabeya (1951, 1952) and Kamat (1953). It may also be observed that this method of derivation is considerably simpler than those adopted by Nair or Lomnicki for deriving the variance of g.

Let $x_i (i = 1, 2, \ldots, n)$ denote a sample from a normal population, $N(\mu, \sigma)$. Then if we denote $z_{ij} = x_i - x_j$, z_{ij} is $N(0, \sqrt{2}\sigma)$ and $\rho(z_{ij}, z_{ik}) = \frac{1}{2}$. For the evaluation of the first three moments of g, we require the following expectations which are readily found by using the formulae for absolute moments (see, for example, Kamat, 1953, pp. 26–7):

$$\left.\begin{aligned}
&\mathscr{E}(|z_{ij}|) = \frac{2\sigma}{\sqrt{\pi}}, \quad \mathscr{E}(z_{ij}^2) = 2\sigma^2, \quad \mathscr{E}(|z_{ij}||z_{ik}|) = \frac{\sigma^2}{\pi}(2\sqrt{3} + \tfrac{1}{3}\pi) \\
&\mathscr{E}(|z_{ij}|^3) = \frac{8\sigma^3}{\sqrt{\pi}}, \quad \mathscr{E}(|z_{ij}|^2|z_{ik}|) = \frac{5\sigma^3}{\sqrt{\pi}}, \\
&\mathscr{E}(|z_{ij}||z_{ik}||z_{il}|) = \frac{8\sigma^3}{\pi^{\frac{3}{2}}}\left(\frac{1}{\sqrt{2}} + \frac{3}{4}\sin^{-1}\tfrac{1}{3}\right), \\
&\mathscr{E}(|z_{ij}||z_{ik}||z_{jk}|) = \frac{3\sigma^3}{\sqrt{\pi}}, \\
&\mathscr{E}(|z_{ij}||z_{jk}||z_{kl}|) = \frac{8\sigma^3}{\pi^{\frac{3}{2}}}\left(\frac{1}{\sqrt{2}} - \tfrac{1}{4}\sin^{-1}\tfrac{1}{3} + \sin^{-1}\frac{1}{\sqrt{3}}\right), \\
&\mathscr{E}(|z_{ij}||z_{ik}||z_{ip}|) = \frac{8\sigma^3}{\pi^{\frac{3}{2}}}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{12}\right).
\end{aligned}\right\} \tag{2}$$

To find the moments of g the following expectations have to be evaluated:

$$\left.\begin{aligned}
\mathscr{E}(g) &= \frac{2}{n(n-1)}\mathscr{E}(\Sigma|z_{ij}|), \\
\mathscr{E}(g^2) &= \frac{4}{n^2(n-1)^2}\mathscr{E}[\Sigma z_{ij}^2 + 2(\Sigma|z_{ij}||z_{ik}| + \Sigma|z_{ij}||z_{kl}|)], \\
\mathscr{E}(g^3) &= \frac{8}{n^3(n-1)^3}\mathscr{E}[\Sigma|z_{ij}^3| + 3(\Sigma z_{ij}^2|z_{ik}| + \Sigma z_{ij}^2|z_{kl}|) + 6(\Sigma|z_{ij}||z_{ik}||z_{il}| + \Sigma|z_{ij}||z_{ik}||z_{jk}| \\
&\qquad + \Sigma|z_{ij}||z_{jk}||z_{kl}| + \Sigma|z_{ij}||z_{ik}||z_{ip}| + \Sigma|z_{ij}||z_{kl}||z_{pq}|)],
\end{aligned}\right\} \tag{3}$$

where i, j, k, l, p, q are all different and the summation symbols denote summation over all of them with the obvious limitations imposed by the definition (1).

It can be shown that the number of terms in the sums of the various types occurring in (3) are as follows:

Type of sum	*Number of terms*
$\|z_{ij}\|\|z_{ik}\|$	$\frac{1}{2}n(n-1)(n-2)$,
$\|z_{ij}\|\|z_{kl}\|$	$\frac{1}{8}n(n-1)(n-2)(n-3)$,
$z_{ij}^2\|z_{ik}\|$	$n(n-1)(n-2)$,
$z_{ij}^2\|z_{kl}\|$	$\frac{1}{4}n(n-1)(n-2)(n-3)$,
$\|z_{ij}\|\|z_{ik}\|\|z_{il}\|$	$\frac{1}{6}n(n-1)(n-2)(n-3)$,
$\|z_{ij}\|\|z_{ik}\|\|z_{jk}\|$	$\frac{1}{6}n(n-1)(n-2)$,
$\|z_{ij}\|\|z_{jk}\|\|z_{kl}\|$	$\frac{1}{6}n(n-1)(n-2)(n-3)$,
$\|z_{ij}\|\|z_{ik}\|\|z_{ip}\|$	$\frac{1}{4}n(n-1)(n-2)(n-3)(n-4)$,
$\|z_{ij}\|\|z_{kl}\|\|z_{pq}\|$	$\frac{1}{48}n(n-1)(n-2)(n-3)(n-4)(n-5)$.

(4)

Substituting in (3) the values of expectations given in (2) and noting (4), after some simplification, we have the following first three moments of g:

$$\mu_1 = \frac{2\sigma}{\sqrt{\pi}} = (1{\cdot}128379)\,\sigma, \tag{5}$$

$$\begin{aligned}\mu_2 &= \frac{4\sigma^2}{n(n-1)\,\pi}\{(2\sqrt{3}-4+\tfrac{1}{3}\pi)\,n+(6-4\sqrt{3}+\tfrac{1}{3}\pi)\}\\ &= \frac{\sigma^2}{n(n-1)}\{(0{\cdot}651006)\,n+0{\cdot}151508\}\\ &\sim \frac{0{\cdot}651006}{n}\,\sigma^2 \quad \text{for large } n.\end{aligned} \tag{6}$$

$$\begin{aligned}\mu_3 &= \frac{8\sigma^3}{n^2(n-1)^2\pi^{\frac{3}{2}}}\{(16\sqrt{2}+40-36\sqrt{3})\,n^2+(144\sqrt{3}-136-80\sqrt{2})\,n+(96\sqrt{2}+120-144\sqrt{3}-2\pi)\}\\ &= \frac{\sigma^3}{n^2(n-1)^2}\{(0{\cdot}393063)\,n^2+(0{\cdot}399735)\,n+0{\cdot}094822\}\\ &\sim \frac{(0{\cdot}393063)\,\sigma^3}{n^2} \quad \text{for large } n.\end{aligned} \tag{7}$$

For $n = 3$ we know that $g = \frac{1}{3}w$, where w is the range. It is easily verified that $\mu_3(g)$ given above checks up with the third moment of w given, for instance, by Hartley & Pearson (1951).

Although it is not possible to say much about the distribution of g on the strength of its first three moments only, the following values of $\sqrt{(\mu_3)}/\mu_1$ and β_1 suggest that it may be close to the χ-distribution. This is confirmed by the χ-approximation to the distribution of the range w for $n = 3$, since in this case $g = \frac{1}{3}w$. The fourth column of the table gives for comparison the values of β_1 for the χ-distribution having the same $\sqrt{(\mu_3)}/\mu_1'$.

n	$\sqrt{(\mu_3)}/\mu_1'$	β_1	β_1 for χ-distribution
5	0·3657	0·1797	0·1672
10	0·2411	0·0708	0·0648
15	0·1926	0·0436	0·0398
20	0·1650	0·0315	0·0286

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A method of systematic sampling based on order properties*

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1. INTRODUCTION

In some theoretical work, particularly in non-parametric inference, the exact probability distributions of certain statistics are so complicated, at least for small samples, that we have to resort to sampling experiments in the laboratory to derive some constants of these distributions. In this paper the method of systematic sampling, which is often employed in surveys and censuses, is utilized to improve upon the method of random samples. In such systematic methods a 'grid' or 'cluster' of points covering the entire population is needed, the randomness entering into the choice of this grid or cluster. It is found that the order properties of samples may be used to determine such a cluster of points.

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One disadvantage lies in the greater difficulty of determining the sampling errors involved. When an idea of the sampling error is not available, the method can be safely used only when there is some external check on the resulting estimates. It has not been possible to evaluate the sampling errors of the method proposed in this paper, but some check is attempted by using some known theoretical results.

2. Systematic sampling from a bivariate population

We consider first the problem of finding the probabilities of the 120 possible rankings of five from a bivariate normal population. In another paper (Sundrum, 1953, submitted to *Biometrika*) these probabilities are used to determine the third and fourth moments of Kendall's rank correlation coefficient in the non-null case. As it appeared extremely difficult to evaluate these probabilities theoretically, it was decided to use estimates based on sampling experiments.

The usual method based on random samples of constructing samples of five from a correlated normal population is to take independent sets, say of five X's and five Y's, from Wold's *Tables of Random Normal Deviates*, and then for each set to find the quantity $Z = \rho X + \sqrt{(1-\rho^2)}\, Y$; then the resulting sets of X and Z may be considered a sample from a bivariate normal population with correlation ρ. The extent to which the sets of X and Z represent samples from a correlated normal population depends on the extent to which the original sets of X and Y represent samples from an independent normal population.

Now, if samples of five bivariate observations of X's and Y's are drawn from an independent normal population, and the X's are arranged in order of magnitude, then it is well known that the 120 possible rankings of the Y's are equally probable. In order to get better estimates for the correlated population, we therefore adopt a method which *ensures* that the samples of correlated observations are constructed from samples of independent observations, when these independent observations are made to conform *exactly* with the condition of equi-probability of the 120 rankings. (In the following a set of X's and Y's will be referred to only by the ranking of the Y's when the X's are arranged in order of magnitude.) The method used is as follows: five X's and five Z's are drawn from Wold's *Tables*. The five Z's are then permuted in the $5! = 120$ possible ways. Each of these sets associated with the set of five X's represents a sample from an independent normal population. If, then, we calculate for each of the 120 bivariate observations the quantities $Y = X + Z'$ (where Z' represents one of the 120 permutations), we get 120 sets of five Y's each, which when associated with the X's may be considered as samples from a bivariate normal population with correlation $\rho = 1/\sqrt{2}$.

To illustrate the method, consider an example with three bivariate observations. From the first six numbers of Wold's *Tables*, write

X	−0·84	1·37	−0·18
Z	0·35	2·82	2·12

The $3! = 6$ samples from an independent population are then

X	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
−0·84	0·35	0·35	2·12	2·12	2·82	2·82
−0·18	2·12	2·82	0·35	2·82	0·35	2·12
1·37	2·82	2·12	2·82	0·35	2·12	0·35

Then the samples from a correlated normal population are obtained by calculating the values of $Y_i = X + Z_i$, thus:

X	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
−0·84	−0·49	0·49	1·28	1·28	1·98	1·98
−0·18	1·94	2·64	0·17	2·64	0·17	1·94
1·37	4·19	3·49	4·19	1·72	3·49	1·72

In the same manner, we can also obtain six samples by taking the permutations of the X's. Thus, with a sample of five bivariate observations, we can generate 240 sets of five bivariate observations each, representing samples from a correlated normal population.

In practice, a systematic procedure may be adopted. For the above example, we construct a 3×3 table as follows:

Z \ X	−0·84	−0·18	1·37
0·35	−0·49	0·17	1·72
2·12	1·28	1·94	3·49
2·82	1·98	2·64	4·19

The body of the table is filled in by adding the values of X and Z corresponding to each cell; e.g. in the first cell on the top left corner we put in $-0{\cdot}84+0{\cdot}35=-0{\cdot}49$. We then get a set of three values, correlated with X, by taking elements from the first, second and third columns, such that no two are in the same row. Similarly, a set of three values, correlated with Y, is obtained by taking elements from the first, second and third rows such that no two are in the same column. This can be done fairly quickly with some practice and may even be mechanized.

This method was used to obtain samples from a bivariate normal population with correlation $\rho = 1/\sqrt{2}$. Fifty bivariate observations of five X's and five Z's were taken from p. 33 of Wold's *Tables*; each of these sets was used to generate 240 observations from the correlated population, giving in all 12,000 bivariate observations. When these 12,000 observations are classified, according to the ranking of the Y's, we obtain the following distribution:

12345	1,044	21345	631	31245	299	41235	123	51234	31
12354	554	21354	274	31254	124	41253	42	51243	25
12435	719	21435	404	31425	161	41325	80	51324	24
12453	282	21453	154	31452	57	41352	29	51342	15
12534	339	21534	132	31524	46	41523	32	51423	16
12543	242	21543	121	31542	32	41532	20	51432	6
13245	631	23145	301	32145	233	42135	92	52134	26
13254	381	23154	133	32154	124	42153	28	52143	16
13425	345	23415	131	32415	110	42315	67	52314	17
13452	129	23451	26	32451	29	42351	10	52341	4
13524	183	23514	53	32514	62	42513	9	52413	11
13542	95	23541	21	32541	24	42531	11	52431	5
14235	416	24135	158	34125	41	43125	63	53124	16
14253	146	24153	38	34152	14	43152	18	53142	5
14325	224	24315	113	34215	52	43215	65	53214	20
14352	95	24351	27	34251	13	43251	15	53241	7
14523	85	24513	16	34512	9	43512	8	53412	7
14532	54	24531	21	34521	6	43521	10	53421	4
15234	179	25134	42	35124	21	45123	19	54123	14
15243	107	25143	27	35142	13	45132	9	54132	5
15324	120	25314	29	35214	29	45213	5	54213	6
15342	58	25341	12	35241	12	45231	5	54231	3
15423	62	25413	9	35412	9	45312	6	54312	4
15432	39	25431	14	35421	4	45321	5	54321	2
				Total	12,000				

From each set of five bivariate observations, we can draw five sets of four bivariate observations, which may also be considered samples from the same correlated population. For example, from the sample represented by (15423), by omitting one observation at a time and re-ranking the remaining observations, we get the samples represented by (4312), (1423), (1423), (1432) and (1432). Thus, we can derive from the above distribution, the distribution of 60,000 observations with four bivariate points each. The distribution of these is as follows:

1234	12,891	2134	6740	3124	2733	4123	847
1243	6,588	2143	3153	3142	944	4132	511
1324	7,571	2314	2788	3214	2129	4213	586
1342	2,825	2341	748	3241	613	4231	292
1423	3,207	2413	1017	3412	395	4312	322
1432	1,979	2431	570	3421	313	4321	238
			Total	60,000			

We can continue the same method to obtain the distribution of rankings of three observations from the above distribution of rankings of four. We then get:

123	106,510
132	48,096
213	47,946
231	13,892
312	14,362
321	9,194
Total	240,000

At this stage we can check the 'quality' of this sample by the correspondence of the above distribution with some theoretical values. I have shown elsewhere that the probabilities of the six rankings of three from a normally correlated population are given by

$$\Pr\{123\} = \frac{3}{8}\left\{\frac{4}{9}+\frac{4}{\pi}\sin^{-1}\rho-\frac{4}{\pi}\sin^{-1}\tfrac{1}{2}\rho+\frac{4}{\pi^2}(\sin^{-1}\rho)^2-\frac{4}{\pi^2}(\sin^{-1}\tfrac{1}{2}\rho)^2\right\}$$

$$\Pr\{132\} = \Pr\{213\} = \frac{3}{4}\left\{\frac{2}{9}-\frac{1}{\pi}\sin^{-1}\rho+\frac{3}{\pi}\sin^{-1}\tfrac{1}{2}\rho-\frac{1}{\pi^2}(\sin^{-1}\rho)^2+\frac{1}{\pi^2}(\sin^{-1}\tfrac{1}{2}\rho)^2\right\}$$

$$\Pr\{231\} = \Pr\{312\} = \frac{3}{4}\left\{\frac{2}{9}+\frac{1}{\pi}\sin^{-1}\rho-\frac{3}{\pi}\sin^{-1}\tfrac{1}{2}\rho-\frac{1}{\pi^2}(\sin^{-1}\rho)^2+\frac{1}{\pi^2}(\sin^{-1}\tfrac{1}{2}\rho)^2\right\}$$

$$\Pr\{321\} = \frac{3}{8}\left\{\frac{4}{9}-\frac{4}{\pi}\sin^{-1}\rho+\frac{4}{\pi}\sin^{-1}\tfrac{1}{2}\rho+\frac{4}{\pi^2}(\sin^{-1}\rho)^2-\frac{4}{\pi^2}(\sin^{-1}\tfrac{1}{2}\rho)^2\right\}.$$

Evaluating these expressions for the case of $\rho = 1/\sqrt{2}$ we can compare with the relative frequencies of these rankings actually achieved in the sample. The results are as follows:

	Theoretical	Sample
123	0·4431	0·4438
132	0·2009	0·2004
213	0·2009	0·1998
231	0·0585	0·0579
312	0·0585	0·0598
321	0·0381	0·0383

The correspondence appears to be satisfactory.

Incidentally, from the distribution of rankings of five, we may derive the small sample distribution of Kendall's rank correlation coefficient from this population. The distribution is as follows:

Distribution of t ($n = 5$) from a normally correlated population $\rho = 1/\sqrt{2}$

t	Frequency
1·0	1044
0·8	2535
0·6	3041
0·4	2452
0·2	1518
0·0	798
−0·2	390
−0·4	149
−0·6	55
−0·8	16
−1·0	2
Total	12,000

The mean and variance of this distribution are 0·4986 (0·50) and 0·10818 (0·10992) respectively, the figures in brackets being the known theoretical values.

In the same manner, we may derive the distribution of the Spearman rank correlation coefficient in samples of five from such a population. The distribution is as follows:

Distribution of r_s ($n = 5$) from a normally correlated population with $\rho = 1/\sqrt{2}$

r_s	Frequency	r_s	Frequency
1·0	1044	−0·1	235
0·9	2535	−0·2	116
0·8	1059	−0·3	142
0·7	1982	−0·4	55
0·6	1242	−0·5	67
0·5	893	−0·6	55
0·4	562	−0·7	37
0·3	896	−0·8	18
0·2	320	−0·9	16
0·1	505	−1·0	2
0·0	219		
		Total	12,000

The mean and variance of r_s from this distribution are 0·5938 and 0·12365 respectively; the theoretical value of the mean, from Moran's formula, is 0·595.

A final assessment of the value of this method must of course await further investigation on the standard errors involved, and the speed with which each method of sampling can be carried out. Clearly the 12,000 values of, say, r_s obtained by 240 recombinations of each of the fifty independent bivariate samples are 'worth' more than fifty independent values of r_s; but they would probably provide less information than we should obtain by calculating one value of r_s from each of 12,000 independent bivariate samples. Thus if the same accuracy is obtained by drawing N independent samples and calculating one value of the statistic from each, as by drawing n samples each providing m values of the statistic by recombination, the practical problem is to determine which procedure involves the smaller expenditure of effort.

3. Application to the univariate case

The above method of systematic sampling may also be applied to some univariate problems. For example, the sampling distribution of many distribution-free statistics for testing the homogeneity of two samples cannot be determined simply under certain non-null cases. Then we may use a systematic procedure of sampling experiments to obtain such distributions. Let the problem be one of determining the sampling distribution of some statistic based on ranks alone to test the homogeneity of two samples of mX's and nY's, when the X's and Y's are drawn from populations which differ only in location. Then a laboratory sample is obtained by taking mX's and nZ's from identical populations of the specified type, and constructing a sample of nY's by adding, say d, to each of the nZ's. The extent to which the samples of X's and Y's represent the populations with different location depends on the extent to which the X's and Z's represent the identical populations. In order to ensure that the X's and Z's do represent samples from identical populations, they should satisfy the requirement that all the $\binom{m+n}{m}$ ways of partitioning a set of $(m+n)$ observations into two sets of m and n respectively should be equally probable. In order to achieve this exactly, the following suggestion is made. We take a set of $(m+n)$ observations and divide them into two sets of m and n in all the $\binom{m+n}{m}$ possible ways. From each of these sets the n observations are transformed into the second sample of nY's by adding the quantity d to each of them. Such samples may then be used to study the distribution of such statistics as Wilcoxon's two-sample test under alternative hypotheses.

4. Conclusion

In using this method, the practical task is very much simplified because of the systematic nature of the sampling. However, it does not seem practicable to use this method for any large sample, for the task of enumerating all the possible permutations would be prohibitive. The main use of this method seems to be to estimate constants of certain distributions, rather than the distributions themselves.

A note on ordered least-squares estimation

BY F. DOWNTON, *University of Liverpool*

The parameters μ, σ of a distribution having the form $f\{(x-\mu)/\sigma\}$ may be estimated by applying the method of least squares to ordered observations, where ordering is according to magnitude. The general theory was given in a paper by Lloyd (1952). It was there shown that the 'ordered' estimate $\hat{\mu}$ of the expectation μ had a variance never exceeding that of the sample mean; and, in the case of symmetric distributions, necessary and sufficient conditions were obtained for the variance of $\hat{\mu}$ to be strictly less than that of the sample mean.

In this note these conditions have been extended to include the unsymmetrical case.

Let $x_1, x_2, \ldots, x_n$ be a sample of n independent observations on a continuous variate X whose distribution has the above form. The possibility of repetitions among the x's then has zero probability. We may therefore, with probability one, write

$$x_{(1)} < x_{(2)} < x_{(3)} < \ldots < x_{(n)} \tag{1}$$

for the ordered observations $x_{(r)}$.

Let

$$z_r = (x_r - \mu)/\sigma, \quad z_{(r)} = (x_{(r)} - \mu)/\sigma,$$

be the reduced observations, unordered and ordered respectively. They have parameter-free distributions, so that their moments may be regarded as known.

Clearly,

$$\mathscr{E}(z_r) = 0 \quad (r = 1, 2, 3, \ldots, n). \tag{2}$$

Let $$\mathscr{E}(z_{(r)}) = \alpha_r, \quad \operatorname{cov}(z_{(r)}, z_{(s)}) = w_{rs} \quad (r, s = 1, 2, 3, \ldots, n).$$

Let $\boldsymbol{\alpha}$ denote the $(n \times 1)$ vector of the α_r, $\mathbf{w}$ the symmetric, positive-definite $(n \times n)$ matrix of the w_{rs}, $\mathbf{1}$ an $(n \times 1)$ vector of 1's, and $\mathbf{x}$ the $(n \times 1)$ vector of the $x_{(r)}$.

We note for future use that $\mathbf{1}'\boldsymbol{\alpha} = 0$, that $\boldsymbol{\alpha} \neq \mathbf{0}$, and that $\mathbf{1}'\mathbf{w}\mathbf{1} = n$. The first of these results follows from the fact that the z_r and the $z_{(r)}$ are simply permutations of the same set of numbers; hence $\Sigma z_r = \Sigma z_{(r)}$ and

$$\mathbf{1}'\boldsymbol{\alpha} = \Sigma\alpha_r = \Sigma\mathscr{E}(z_{(r)}) = \mathscr{E}(\Sigma z_{(r)}) = \mathscr{E}(\Sigma z_r) = \Sigma\mathscr{E}(z_r) = 0.$$

A similar argument shows that $\mathbf{1}'\mathbf{w}\mathbf{1} = n$.

The second result follows from (1). For with $r < s$, we have $z_{(r)} < z_{(s)}$ and, taking expectations, $\alpha_r < \alpha_s$. Therefore $\alpha_r \neq \alpha_s$ and hence $\boldsymbol{\alpha} \neq \mathbf{0}$.

The 'ordered' estimate $\hat{\mu}$, of μ, and its variance, are given by

$$\hat{\mu} = \boldsymbol{\alpha}'\mathbf{w}^{-1}(\boldsymbol{\alpha}\mathbf{1}' - \mathbf{1}\boldsymbol{\alpha}')\,\mathbf{w}^{-1}\mathbf{x}/\Delta,$$

$$\operatorname{var}(\hat{\mu}) = \boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha}\sigma^2/\Delta,$$

where $$\Delta = (\mathbf{1}'\mathbf{w}^{-1}\mathbf{1})(\boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha}) - (\mathbf{1}'\mathbf{w}^{-1}\boldsymbol{\alpha})^2.$$

Lloyd showed that when the distribution is symmetric, $\mathbf{1}'\mathbf{w}^{-1}\boldsymbol{\alpha} = 0$, which considerably simplifies these expressions. He also showed that a necessary and sufficient condition for the variance of $\hat{\mu}$ to be strictly smaller than σ^2/n is that $\mathbf{w}\mathbf{1} \neq \mathbf{1}$, or, equivalently, that $\operatorname{var}(\hat{\mu}) = \sigma^2/n$ if and only if $\mathbf{w}\mathbf{1} = \mathbf{1}$. He showed further that when the variance of $\hat{\mu}$ equals that of the sample mean, $\hat{\mu}$ itself coincides with the sample mean.

In the general case, where no symmetry assumptions are made we first note that both $\mathbf{w}$ and $\mathbf{w}^{-1}$ are positive-definite and symmetric and may therefore be expressed in the form

$$\mathbf{w} = \mathbf{t}\mathbf{t}' \quad \text{and} \quad \mathbf{w}^{-1} = (\mathbf{t}^{-1})'\mathbf{t}^{-1},$$

where $\mathbf{t}$ is a lower triangular matrix.

Consider any two $(n \times 1)$ vectors $\mathbf{b}$ and $\mathbf{c}$.

Then

$$\mathbf{b}'\mathbf{w}\mathbf{b} = \mathbf{b}'\mathbf{t}\mathbf{t}'\mathbf{b} = \mathbf{h}'\mathbf{h} = \Sigma h_i^2, \quad \text{say},$$

where $\mathbf{h} = \mathbf{t}'\mathbf{b}$.

Also

$$\mathbf{c}'\mathbf{w}^{-1}\mathbf{c} = \mathbf{c}'(\mathbf{t}^{-1})'\mathbf{t}^{-1}\mathbf{c} = \mathbf{k}'\mathbf{k} = \Sigma k_i^2, \quad \text{say},$$

where $\mathbf{k} = \mathbf{t}^{-1}\mathbf{c}$.

Now by the Cauchy-Schwarz inequality we have

$$(\Sigma h_i^2)(\Sigma k_i^2) \geqslant (\Sigma h_i k_i)^2, \tag{3}$$

and the necessary and sufficient condition for equality is that

$$h_i = \lambda k_i \quad \text{for some constant } \lambda \text{ and for all } i. \tag{4}$$

In matrix form (3) and (4) become

$$(\mathbf{b}'\mathbf{w}\mathbf{b})(\mathbf{c}'\mathbf{w}^{-1}\mathbf{c}) \geqslant (\mathbf{b}'\mathbf{c})^2, \tag{5}$$

with

$$\mathbf{b} = \lambda \mathbf{w}^{-1}\mathbf{c} \quad \text{as the condition for equality.} \tag{6}$$

Put $\mathbf{b} = \mathbf{w}^{-1}\mathbf{1} - \mathbf{1}$ and $\mathbf{c} = \boldsymbol{\alpha}$. Then (5) becomes

$$(\mathbf{1}'\mathbf{w}^{-1}\mathbf{1} - 2\mathbf{1}'\mathbf{1} + \mathbf{1}'\mathbf{w}\mathbf{1})(\boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha}) \geqslant (\mathbf{1}'\mathbf{w}^{-1}\boldsymbol{\alpha} - \mathbf{1}'\boldsymbol{\alpha})^2.$$

Now

$$\mathbf{1}'\mathbf{1} = \mathbf{1}'\mathbf{w}\mathbf{1} = n \quad \text{and} \quad \mathbf{1}'\boldsymbol{\alpha} = 0.$$

Thus

$$(\mathbf{1}'\mathbf{w}^{-1}\mathbf{1} - n)(\boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha}) \geqslant (\mathbf{1}'\mathbf{w}^{-1}\boldsymbol{\alpha})^2,$$

or

$$\{(\mathbf{1}'\mathbf{w}^{-1}\mathbf{1})(\boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha}) - (\mathbf{1}'\mathbf{w}^{-1}\boldsymbol{\alpha})^2\}/(\boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha}) = \sigma^2/\mathrm{var}(\hat{\mu}) \geqslant n,$$

since $\boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha}$ is essentially positive. Thus $\mathrm{var}(\hat{\mu}) \leqslant \sigma^2/n$; and the necessary and sufficient condition for equality is, from (6), that

$$\mathbf{w}\mathbf{1} = \mathbf{1} - \lambda\boldsymbol{\alpha} \tag{7}$$

for some scalar constant λ. When this condition is satisfied not only is the variance of $\hat{\mu}$ equal to that of the sample mean, but $\hat{\mu}$ itself necessarily *is* the sample mean.

For, pre-multiplying (7) by $\boldsymbol{\alpha}'\mathbf{w}^{-1}$, $\mathbf{1}'\mathbf{w}^{-1}$ and $\mathbf{w}^{-1}$ we have

$$\boldsymbol{\alpha}'\mathbf{w}^{-1}\mathbf{1} - \lambda\boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha} = \boldsymbol{\alpha}'\mathbf{1} = 0, \tag{8}$$

$$\mathbf{1}'\mathbf{w}^{-1}\mathbf{1} - \lambda\mathbf{1}'\mathbf{w}^{-1}\boldsymbol{\alpha} = \mathbf{1}'\mathbf{1} = n, \tag{9}$$

and

$$\mathbf{w}^{-1}\mathbf{1} - \lambda\mathbf{w}^{-1}\boldsymbol{\alpha} = \mathbf{1}, \tag{10}$$

respectively.

Now

$$\begin{aligned}\Delta &= (\boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha})(\mathbf{1}'\mathbf{w}^{-1}\mathbf{1}) - (\mathbf{1}'\mathbf{w}^{-1}\boldsymbol{\alpha})^2 \\ &= \boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha}(\mathbf{1}'\mathbf{w}^{-1}\mathbf{1} - \lambda\mathbf{1}'\mathbf{w}^{-1}\boldsymbol{\alpha}) \quad \text{from (8)} \\ &= n\boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha} \quad \text{from (9).}\end{aligned} \tag{11}$$

Also

$$\begin{aligned}\hat{\mu} &= (\boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha}\mathbf{1}' - \boldsymbol{\alpha}'\mathbf{w}^{-1}\mathbf{1}\boldsymbol{\alpha}')\,\mathbf{w}^{-1}\mathbf{x}/\Delta \\ &= \boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha}(\mathbf{1}'\mathbf{w}^{-1} - \lambda\boldsymbol{\alpha}'\mathbf{w}^{-1})\,\mathbf{x}/\Delta \quad \text{from (8)} \\ &= \boldsymbol{\alpha}'\mathbf{w}^{-1}\boldsymbol{\alpha}\mathbf{1}'\mathbf{x}/\Delta \quad \text{from (10),}\end{aligned}$$

whence $\hat{\mu} = \mathbf{1}'\mathbf{x}/n$, which is the sample mean.

I am grateful to Dr E. H. Lloyd for an extremely stimulating correspondence on this subject.

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A note on the evaluation of the multivariate normal integral

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1. A problem which arises frequently is concerned with the evaluation of the multivariate normal integral. Thus if $x_1, x_2, \ldots, x_n$ are normal variables, all correlated, we often need

$$P\{x_1 > 0, x_2 > 0, \ldots, x_n > 0\}.$$

Little work of practical use has appeared since the papers of Moran (1948) and Kendall (1941). It is the purpose of this note to challenge others to the solution of the problem and to summarize the position, as the writer sees it, at this present time.

2. Let the event E_i consist in the random variable x_i being positive. Thus

$$P\{x_i>0\} = P\{E_i\} = \tfrac{1}{2}.$$

For two variables x_1 and x_2 we have the well-known result of Sheppard (1898),

$$P\{x_1>0, x_2>0\} = P\{E_1E_2\} = \frac{1}{2}\left[1-\frac{\cos^{-1}\rho_{12}}{\pi}\right],$$

where ρ_{12} is the correlation between x_1 and x_2. This result is simply achieved by an appeal to geometry as is also the result for three variables. With three variables the probability ellipsoid can be transformed into a sphere with the original co-ordinate planes becoming three planes through the origin. It is easy to show for three (or more) dimensions that the angle between the ith and jth transformed co-ordinate planes is just $\cos^{-1}\rho_{ij}$, where ρ_{ij} is the correlation between the original variables x_i and x_j. The required probability for three (or more) variables is just the solid angle in three or more dimensions when the angles between the planes containing the solid angle are given. An exact expression for this solid angle appears to be known for three dimensions only.

3. A curious recurrence relation for the required probabilities is possible. This relation has its analogue in many branches of mathematics; we may quote, for example, any text-book on multi-dimensional geometry. It was also partially explored in relation to an n-dimensional integral by Schaefli (1857). The theorem usually ascribed to Boole is as follows:

$$P\{E_1+E_2+\ldots+E_n\} = \sum_i P\{E_i\} - \sum_{i<j} P\{E_iE_j\} + \sum_{i<j<l} P\{E_iE_jE_l\} - \ldots + (-1)^{n-1}P\{E_1E_2\ldots E_n\}.$$

Now for the problem considered

$$P\{E_1+E_2+\ldots+E_n\} = 1-P\{\bar{E}_1\bar{E}_2\ldots\bar{E}_n\} = 1-P\{E_1E_2\ldots E_n\},$$

by the symmetry of the multivariate normal distribution. Hence for n *odd* we have

$$P\{E_1E_2\ldots E_n\} = \tfrac{1}{2}[1-\sum_i P\{E_i\} + \sum_{i<j} P\{E_iE_j\} - \ldots \text{etc.}],$$

the last term on the right-hand side being the sum of the joint probabilities of the E's taken $n-1$ at a time. The result for $n = 3$ is immediate:

$$P\{E_1E_2E_3\} = \frac{1}{2}\left[1-\frac{3}{2}+\frac{3}{2}-\frac{\cos^{-1}\rho_{12}}{2\pi}-\frac{\cos^{-1}\rho_{13}}{2\pi}-\frac{\cos^{-1}\rho_{23}}{2\pi}\right]$$

$$= \frac{1}{4\pi}[2\pi-\cos^{-1}\rho_{12}-\cos^{-1}\rho_{13}-\cos^{-1}\rho_{23}].$$

For n even we get an identity.

4. From the recurrence relation we can deduce the results for $n = 5$ given that for $n = 4$, but it does not seem possible to obtain a recurrence relation which will enable us to make the step from 3 to 4. Moran, following Kendall, gave an expansion for $P\{E_1E_2E_3E_4\}$, but experience has shown that this does not converge fast enough for practical purposes except for ρ_{ij} small. For example, for ρ_{ij} in the neighbourhood of 0·5 at least twenty sets of terms are necessary to obtain reasonable accuracy. Some work has been done towards obtaining a series expansion for ρ_{ij} nearly 0·5 in terms of $(\rho_{ij}-0{\cdot}5)$, but only in certain restricted cases. What is required is a quickly convergent series for ρ_{ij} large and n even. From $n = 4$ we deduce $n = 5$, calculate $n = 6$ from the series, deduce $n = 7$ and so on. The problem is difficult but not unrewarding.

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A graphical method for the analysis of statistical distributions into two normal components

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1. Introduction

Many frequency distributions occur in statistical practice which are probably due to the existence of two or more separate sub-universes within the universe under consideration, each of which is of approximately Normal form; for instance, the heights or weights of English men and women, or the intelligence quotients of professional and artisan engineers would probably have this property. If the sub-universes have different mean values, different variances and contain different numbers of individuals, a great variety of composite distributions will result. The existence of sub-universes is not usually obvious from the bimodal or multi-modal appearance of a frequency curve, since separate modes do not appear unless the separation of the means is considerable; it needs to be about three times the standard deviation of the components, if they are of comparable size. The following problem therefore arises: given a sample (preferably of several thousand individuals), selected from a universe suspected for some reason of having such components, can we determine the most probable nature of the sub-universes? Perhaps most important, is there a quick practical way of doing so?

The 'method of moments' was first used in a theoretical solution of the problem by K. Pearson (1894). Since then, the more efficient 'method of maximum likelihood' has been developed by R. A. Fisher and others, and applied in particular to this problem by C. R. Rao (1948). Rao also gives, in the same paper, a rapid and elegant theoretical solution, by the method of moments, for the case of two components assumed to have equal variances. This depends upon the solution of a cubic equation, and appears to yield quite accurate results.

However, it seems to us that all these contributions suffer from over-complexity and the necessity for lengthy calculations. The methods do indeed give the most accurate results possible from the available sample, but this degree of accuracy seems rather unnecessary in view of the unavoidable inherent errors due to sampling fluctuations and to the assumptions that the true components are normal and have equal variances. Thus there seems to be room for a more rapid, and much less laborious, graphical method.

This paper deals with such a method for the simple case of two components assumed to have equal variances; the variances seem to be very often approximately equal in practice and the assumption permits a much more immediate solution of the problem. The method is based on the method of moments and the use of a 'skewness-kurtosis' diagram.

2. Notation

Let the means of the two component normal populations be μ_1 and μ_2 and let σ be their common standard deviation. Let the subscript 1 be applied to the larger component, and let ρ be the ratio of the number of individuals in the first component to those in the second, so that $\rho \geqslant 1$. Write $\delta = (\mu_2 - \mu_1)/\sigma$ and denote the cumulants of the composite distribution by κ_1, κ_2, etc., where κ_1 is the distance of the mean from some convenient origin. Then it may be shown that

$$\kappa_1 = (\rho\mu_1 + \mu_2)/(\rho+1),$$
$$\kappa_2 = \sigma^2\{(\rho+1)^2 + \rho\delta^2\}/(\rho+1)^2,$$
$$\kappa_3 = \sigma^3\rho(\rho-1)\,\delta^3/(\rho+1)^3,$$
$$\kappa_4 = \sigma^4\rho(\rho^2-4\rho+1)\,\delta^4/(\rho+1)^4.$$

Hence

$$\gamma_1 = \sqrt{\beta_1} = \frac{\kappa_3}{\kappa_2^{\frac{3}{2}}} = \frac{\rho(\rho-1)\,\delta^3}{\{\rho\delta^2 + (\rho+1)^2\}^{\frac{3}{2}}}, \tag{1}$$

$$\gamma_2 = \beta_2 - 3 = \frac{\kappa_4}{\kappa_2^2} = \frac{\rho(\rho^2-4\rho+1)\delta^4}{\{\rho\delta^2 + (\rho+1)^2\}^2}. \tag{2}$$

3. The 'skewness-kurtosis' diagram

The accompanying diagram shows systems of contours for constant ρ and constant δ plotted with γ_1 and γ_2 as rectangular co-ordinates. If $\gamma_1 < 0$, the chart may be entered with $|\gamma_1|$ and γ_2 as co-ordinates, the values of δ being given negative signs. Thus if γ_1 (and κ_3) is positive, the larger component has the lower mean, while if $\gamma_1 < 0$ the reverse is the case.

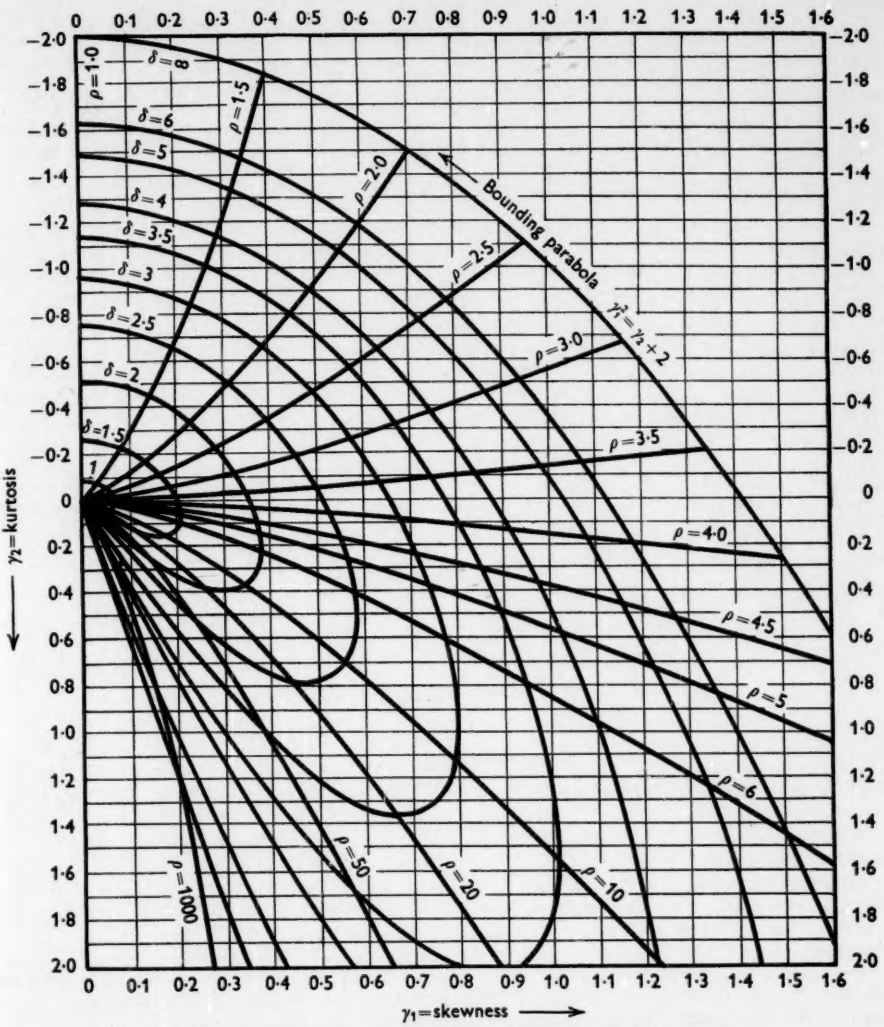

Fig. 1. Skewness-kurtosis (γ_1, γ_2) chart for distributions with two normal components having equal variance.

The lines form a network within the 'bounding parabola' $\gamma_1^2=\gamma_2+2$, so that in this region values of ρ and δ corresponding to any point exist, are unique, and may be estimated. The 'bounding parabola' corresponds to infinite separation of the components and points inside it satisfy the condition

$$\kappa_4+2\kappa_2^2-\kappa_3^2/\kappa_2\geqslant 0,$$

satisfied by all real frequency distributions, so that no real points can lie outside the parabola.

Other interesting properties of the diagram, most of which are of purely theoretical importance, are outlined here:

If the sub-universes are equal in size, $\rho=1, \gamma_1=0$ and γ_2 is negative; the distribution is symmetrical and rather platykurtic, as is also obvious from consideration of the shape of the frequency curve. With

unequal components there is skewness, and platy- or leptokurtosis (i.e. the excess kurtosis is negative or positive) according as ρ is less or greater than $(2+\sqrt{3})$. If $\rho = (2+\sqrt{3})$ there is no excess kurtosis for any separation. As ρ becomes large, γ_1 and γ_2 both decrease towards zero, and with any finite separation tend to the limit zero as ρ tends to infinity (with infinite separation, γ_2 diverges to infinity). With zero separation, trivially, both γ_1 and γ_2 are zero for all values of ρ. The curves of constant separation are a family of cardioids, and with large values of ρ, near the origin, closely approach the straight lines $\gamma_2 = \gamma_1\delta$. The curves of constant ratio ρ are quartic parabolas, $\gamma_1^4/\gamma_2^3 = \rho(\rho-1)^4/(\rho^2-4\rho+1)^3$ (constant).

4. Use of the diagram

To estimate ρ and δ by the method of moments when we have available a sample of n individuals from the composite population, we obtain in the usual way from the k-statistics estimates g_1 and g_2 of γ_1 and γ_2. These may be used as co-ordinates in the chart to obtain approximate values r and d for ρ and δ, by interpolating between the contours. An estimate, s, of σ may then be obtained from

$$s = (r+1)\sqrt{k_2}/\{(r+1)^3 + rd^2\}^{\frac{1}{4}}, \tag{3}$$

while the estimates m_1 and m_2 of μ_1 and μ_2 are given by

$$m_1 = k_1 - ds/(r+1), \quad m_2 = k_1 + rds/(r+1). \tag{4}$$

Here k_1 and k_2 are the first two k-statistics of the composite sample.

5. Practical use of the method

We take as a first example the data used by Rao (1948) for the same purpose, viz. the heights in centimetres of 454 plants, shown in Table 1.

Table 1. *Heights of 454 plants in centimetres*

Central value	8	9	10	11	12	13	14	15	16	17	18	19	20	Total
Frequency	3	9	21	40	59	76	79	69	46	30	13	7	2	454

The k-statistics (calculated by Rao) are:

$$k_1 = -0{\cdot}244 \text{ (about 14 as origin)}, \quad k_2 = 0{\cdot}729,$$
$$k_3 = 4{\cdot}976, \quad k_4 = -5{\cdot}315.$$

Hence
$$g_1 = 0{\cdot}066, \quad g_2 = -0{\cdot}215.$$

Using these values in the chart, we obtain the estimates

$$r = 1{\cdot}4, \quad d = 1{\cdot}4.$$

From equation (3),
$$s = 1{\cdot}836.$$

From equations (4),
$$m_1 = -0{\cdot}244 - 2{\cdot}57/2{\cdot}4 = -1{\cdot}31,$$
$$m_2 = -0{\cdot}244 + 1{\cdot}4 \times 2{\cdot}57/2{\cdot}4 = 1{\cdot}26.$$

The chief source of error in these estimates lies in the difficulty of accurate estimation from the diagram. Assuming a standard error of $\pm 0{\cdot}05$ in the estimation of both r and d, the standard errors of the results due to errors of interpolation may be found from

$$\left.\begin{aligned}
\sigma_{p_1}^2 &= \sigma_r^2/(1+\rho)^4, \quad [p_1 = r/(r+1)],\\
\sigma_s^2 &= \frac{\sigma^4\delta^2}{4\kappa_2^3(\rho+1)^6}\{(\rho-1)^2\delta^2\sigma_r^2 + 4\rho^2(\rho+1)^2\sigma_d^2\},\\
\sigma_{m_1}^2 &= \frac{\sigma^4}{4\kappa_2^3(\rho+1)^6}\{[2(\rho+1)+\delta^2]^2\delta^2\sigma_r^2 + 4(\rho+1)^4\sigma_d^2\},\\
\sigma_{m_2}^2 &= \frac{\sigma^4}{4\kappa_2^3(\rho+1)^6}\{[2(\rho+1)+\rho\delta^2]^2\delta^2\sigma_r^2 + 4\rho^2(\rho+1)^4\sigma_d^2\}.
\end{aligned}\right\} \tag{5}$$

The results of these calculations, together with the theoretical results of Rao by the methods of moments and of maximum likelihood, and the standard errors of the methods due to sampling and assumptions, are given in Table 2.

Table 2

Parameter estimated	Estimate	Estimated value by method of			Standard error of estimation from diagram ($\sigma_r = \sigma_d = 0{\cdot}05$)	Standard error of max. likelihood estimates
		Max. likelihood	Moments	γ_1, γ_2 diagram		
$\rho/(1+\rho)$	p_1	0·567	0·583	0·58	0·009	0·161
μ_1	m_1	−1·345	−1·341	−1·31	0·034	0·049
μ_2	m_2	1·282	1·286	1·26	0·043	0·033
σ	s	1·823	1·816	1·84	0·021	0·049

The reading errors are rather large, but could be reduced if necessary by constructing a large number of contours on as large a scale as was convenient. Also, in some cases the errors would be greater owing to the shape of the diagram, but ρ and δ are usually small ($\rho < 5, 1 < |\delta| < 3$) and can then be estimated with considerable accuracy.

A second example may be taken from the distributions of length of trypanosome strains analysed in some detail by K. Pearson (1914). The second column of Table 3 shows the observed frequency distribution of the length (in microns) of 1000 individuals of *Trypanosoma gambiense* as recorded by Pearson. The moments and moment ratios of the distribution, using our present notation, are given by Pearson (p. 131)* as

$$k_1 = 22{\cdot}1130, \quad g_1 = 0{\cdot}5361,$$

$$k_2 = 14{\cdot}3389, \quad g_2 = -0{\cdot}4156.$$

Entering our chart with these values of g_1 and g_2, we read off the estimates of ρ and δ as

$$r = 2{\cdot}70, \quad d = 3{\cdot}15.$$

Hence we find

$$s = 2{\cdot}202, \quad m_1 = 20{\cdot}238, \quad m_2 = 27{\cdot}175,$$

$$p_1 = r/(1+r) = 0{\cdot}730, \quad p_2 = 1 - p_1 = 0{\cdot}270.$$

The expected frequencies, based on the two normal components with a common standard deviation estimated by s, are shown in the third column of Table 3 and the differences (observed − theory) in the fourth. After combining four groups in each tail, it is found that $\chi^2 = 15{\cdot}28$, with 13 degrees of freedom. The probability of exceeding this value if our theoretical structure were correct is 0·29. The fit is quite good, but it is evident that there are certain systematic differences between theory and observation.

K. Pearson (1914, p. 131) did not assume a common standard deviation for the two components and used his general moment method of fitting (1894) involving the solution of a nonic. He found

$$m_1 = 19{\cdot}8926, \quad s_1 = 2{\cdot}0566; \qquad m_2 = 26{\cdot}2463, \quad s_2 = 2{\cdot}6260;$$

$$p_1 = 0{\cdot}6505, \quad p_2 = 0{\cdot}3495.$$

Using these estimates of the parameters, the theoretical distribution given in the fifth column of Table 3 is obtained, with the differences (observed − theory) of the sixth column. In this case $\chi^2 = 10{\cdot}77$ with 12 degrees of freedom, a value exceeded by chance with a probability of 0·55. The fit has been improved by allowing the two variances to differ, but there is still evidence of some systematic discrepancy, as shown by the run of the signs of differences.

6. Summary

This graphical method of analysis into two components by means of the skewness-kurtosis diagram may be useful in many cases where a simple, rapid determination of the probable nature of the two sub-universes is wanted. It is less efficient and less accurate than existing theoretical methods, but there are always unavoidable errors due to sampling and necessary assumptions about the components. It is

* Pearson gives values of moments, with n as divisor, not k-statistics, but with $n = 1000$ the slight adjustment called for seemed unnecessary.

therefore likely to be of particular value, in contrast to these methods, in producing quick approximations to the components of small samples. Any kind of distribution may be analysed by this method into two normal components, irrespective of whether its character is, or is not, due to their physical existence, since all real points on the diagram lie within the bounding parabola; hence the method cannot offer any confirmation of their physical existence. It may, however, permit us to devise physical and experimental tests to identify their physical nature if their existence seems probable.

Table 3

Central value	Observed frequency	Distribution fitted by diagram	Difference	Pearson's composite distribution	Difference
12	—	0·2		0·1	
13	1	0·6		0·5	
14	— } 10	2·5 } 11·4	−1·4	2·2 } 10·6	−0·6
15	9	8·1		7·8	
16	21	21·2	−0·2	21·6	−0·6
17	56	45·3	10·7	47·4	8·6
18	79	78·9	0·1	82·9	−3·9
19	114	112·3	1·7	115·1	−1·1
20	122	130·5	−8·5	128·0	−6·0
21	110	124·6	−14·6	115·8	−5·8
22	85	98·9	−13·9	89·2	−4·2
23	85	68·8	16·2	65·6	19·4
24	61	48·7	12·3	54·5	6·5
25	47	43·3	3·7	53·3	−6·3
26	49	46·7	2·3	54·3	−5·3
27	47	49·7	−2·7	51·0	−4·0
28	44	45·6	−1·6	42·4	1·6
29	31	34·8	−3·8	30·7	0·3
30	20	21·6	−1·6	19·2	0·8
31	11	11·0	0·0	10·5	0·5
32	4	4·6		4·9	
33	4	1·6		2·0	
34	— } 8	0·4 } 6·7	1·3	0·7 } 7·9	0·1
35	—	0·1		0·1	
Total	1000	1000		1000	

I wish to record my sincere thanks to the Editors of *Biometrika* for their most generous and helpful advice and assistance in preparing the final draft of this paper; to my brother, Dr F. W. Preston, of Butler, Pa., U.S.A., through whose interest and at whose suggestion I originally began investigation of the subject; and to his wife Jane, and to Mr R. R. Lehnerd, of the Preston Laboratories, Butler, for their care and patience in making legible the original script and diagrams.

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A note on regions for tests of kurtosis

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The most common test criterion for kurtosis is b_2, the sample fourth moment divided by the fourth power of the standard deviation; an alternative criterion proposed by Geary (1935) and denoted by a is the ratio of the sample mean deviation to the sample standard deviation. The present author has shown in a paper appearing on p. 318 of this volume that Bartlett's (1937) modification of the Neyman-Pearson L_1 criterion (1931) for testing for the equality of variances is so sensitive to kurtosis that when there is only one observation per group (group means assumed known) this criterion denoted by $M_{.1}$ is of the same order of sensitivity to kurtosis as is b_2 or a.

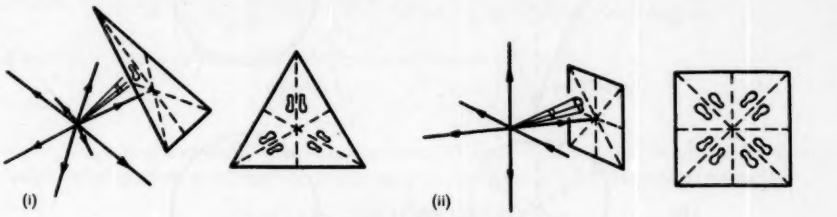
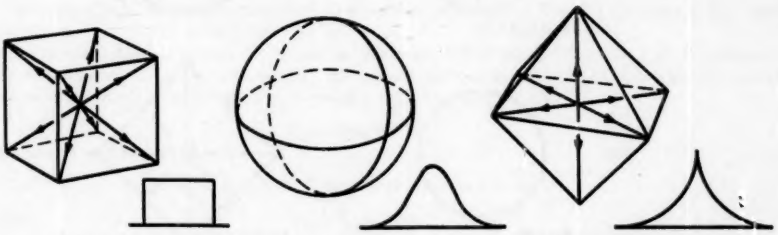


Illustration of symmetry conditions when $N = 3$



(iii) Rectangular (iv) Normal (v) Double exponential

Distributions and probability contours showing directions of accumulation of probability density

Fig. 1. Geometrical properties of tests of kurtosis.

A better understanding of the relations which the various tests for kurtosis have to one another and to $M_{.1}$ is gained by considering the general nature of critical regions for such tests. For simplicity we shall need to assume that the mean is known and equal to zero. We notice first that any criterion to detect kurtosis must be independent of scale; if we reach a certain conclusion from a sample $y_1, y_2, \ldots, y_N$ we must reach the same conclusion from the sample $ky_1, ky_2, \ldots, ky_N$. It follows that the boundary of the critical region in the sample space consists of one or more cones with apices at the origin. Also each observation has equal weight so that if the point $y_1, y_2, \ldots, y_N$ is on the boundary of a critical region then so must the remaining $N! - 1$ points obtained by permuting the order of the values. Finally, since a test of kurtosis should detect departures from normality other than those due to asymmetry of the distribution, the test criterion must be independent of the signs of the observations. Thus the pattern of critical subregions in the region of the space in which the observations are all positive will be repeated in all the remaining $2^N - 1$ regions of the space generated by possible differences in sign. The conditions of symmetry thus imposed are illustrated for $N = 3$ in Fig. 1 (i) and (ii) which shows cross-sections of subsets of cones satisfying these conditions. When the null-hypothesis of normality is true, the probability density in the sample space is constant over the surfaces of N-dimensional hyperspheres with centre at the origin. Consequently if the hypercones of the critical region are such that they include a proportion α of the total surface of such hyperspheres the error of the first kind is controlled at the level α irrespective of the value of the scale parameter.

In addition to controlling this error, we require the test to pick out as often as possible a departure from normal-kurtosis when it occurs. Consider the probability contours corresponding to symmetrical distributions having varying amounts of kurtosis in the case $N = 3$. For the normal distribution the contours are spheres (Fig. 1 (iv)). As γ_2 falls below zero, the contours tend to belly out and for the rectangular distribution ($\gamma_2 = -1{\cdot}2$) the single probability contour is the surface of a cube (Fig. 1 (iii)), the density within the cube being uniform. As γ_2 is made larger than zero the contours tend to sag

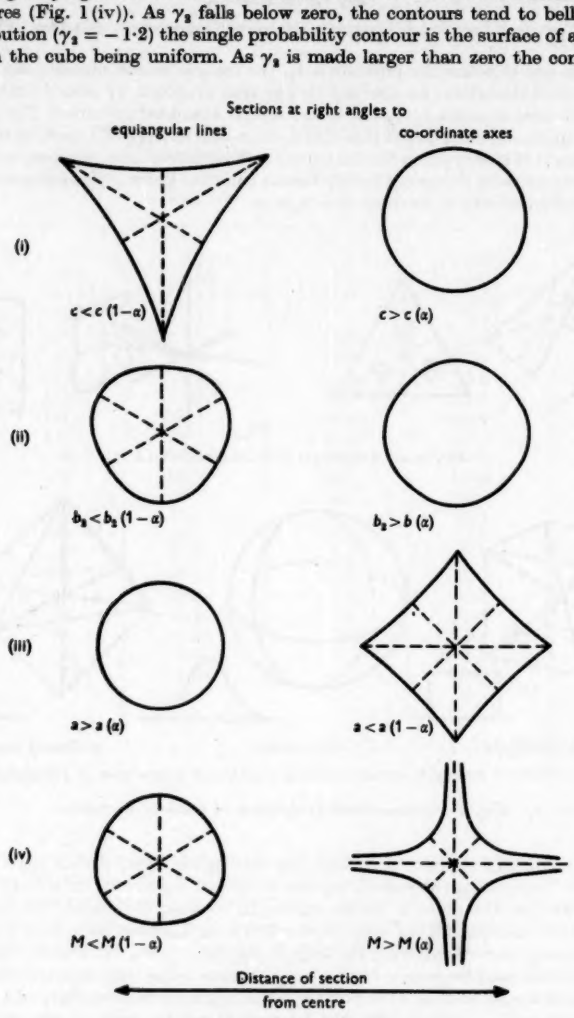


Fig. 2. Sections of critical regions.

inwards till for the double exponential distribution ($\gamma_2 = 3$) the contours are surfaces of regular octahedra (Fig. 1 (v)). In general for platykurtic distributions ($\gamma_2 < 0$) there is an accumulation of probability density along the equiangular lines (and a depletion along the co-ordinate axes), whilst for leptokurtic distributions ($\gamma > 0$) there is an accumulation of probability density along the co-ordinate axes (and a depletion in the direction of the equiangular lines). It follows that (i) a good critical region for testing for platykurtosis would be the 2^N hypercones having apices at the centre and axes along the 2^N equiangular lines, whilst (ii) a good critical region for testing for leptokurtosis would consist of the $2N$ hypercones having their apices along the $2N$ branches of the co-ordinate axes.

The directions of the axes of cones forming such regions when $N = 3$ are shown by arrows in Fig. 1 (iii) and (v) respectively. In particular b_2 and a have critical regions of these types and so has $M_{.1}$. In each case, the particular type which occurs depends on whether we are using the criterion to test for leptokurtosis or for platykurtosis. The regions differ only in the shapes of their cross-section and even these differences are limited to some extent by the symmetry conditions listed above. For the case $N = 3$, sections of cones corresponding to the approximate upper and lower 5 % levels for the test criteria b_2, a and $M_{.1}$ are shown in Fig. 2 (ii), (iii) and (iv). (In order to make clear that the region in Fig. 2 (ii) defined by $b_2 > b_2(\alpha)$ has a non-circular section, it has been necessary slightly to exaggerate the departure from circularity.)

2. It is of some interest to consider kurtosis tests from the point of view of the Neyman-Pearson theory of best critical regions. Suppose that the distribution specified by the alternative hypothesis is a member of the family

$$p(y) = \left\{\frac{2\rho}{q}\,\Gamma\left(\frac{1}{q}\right)\right\}^{-1} \exp -\left\{\frac{|y|}{\rho}\right\}^q \quad (-\infty < y < +\infty, 0 < q < \infty, 0 < \rho < \infty), \tag{1}$$

and q *is given*. This is a symmetrical distribution with mean zero and

$$\kappa_2 = \rho^2 \frac{\Gamma(3/q)}{\Gamma(1/q)}, \quad \gamma_2 = \frac{\Gamma(5/q)\,\Gamma(1/q)}{\{\Gamma(3/q)\}^2} - 3. \tag{2}$$

When $q = 2$, γ_2 is zero and the distribution is normal. When $q > 2$ the distribution is platykurtic and, in particular, when q tends to infinity the distribution tends to the rectangular, for (1) tends to

$$p(y) \begin{cases} = 1/(2\rho) & (y < \rho) \\ = 0 & (y > \rho). \end{cases} \tag{3}$$

When $q < 2$ the distributions are leptokurtic in particular when $q = 1$ the distribution is the double exponential.

The likelihood ratio test criterion (Neyman & Pearson, 1928) for testing the null hypothesis of normality ($q = 2$) against the *single alternative* that the distribution is of the form of the above equation with q equal to some other *specific* value q_0 is readily found to be proportional to

$$\Lambda_{q_0} = (m_{q_0})^{1/q_0}/s, \tag{4}$$

where m_q is the qth absolute moment,

$$m_q = \Sigma\,|y|^q/N, \quad \text{and in particular } s = m_2^{\frac{1}{2}}. \tag{5}$$

The inequality

$$\Lambda_{q_0} < \Lambda_{q_0}(1-\alpha), \tag{6}$$

where $\Lambda(1-\alpha)$ and $\Lambda(\alpha)$ refer respectively to the lower and upper significance points of the criterion, defines a set of cones lying along the co-ordinate axes when $q_0 < 2$ and a set of cones lying along the equiangular lines when $q_0 > 2$. That every such set defines a *best* critical region for the appropriate value of q_0 can be seen from the fact that on every* hyperspherical shell (4) and (6) define the region, independent of scale factors in the null distribution or in the alternative distribution, the boundary of which is given by the fundamental inequality $p_1 \geqslant kp_0$ of Neyman & Pearson (1933). This region on every such shell thus contains for its size the greatest possible concentration of probability density when the appropriate alternative hypothesis is true.

We see that, $a < a(1-\alpha)$ defines the most powerful test when the alternative distribution is the double exponential, for which $\gamma_2 = 3$, also $b_2 < b_2(1-\alpha)$ defines the most powerful test when the alternative distribution is $p(y) = \text{const.} \exp -(y/\rho)^4$ for which $\gamma_2 = -0{\cdot}812$. When the alternative distribution is rectangular ($\gamma_2 = -1{\cdot}2, q = \infty$) the test is defined by $c < c(1-\alpha)$, where

$$c = |y_L|/s \tag{7}$$

and y_L is the observation largest in absolute magnitude. It should be noted that with this criterion we should be testing whether the largest deviate in absolute magnitude were *too small* and the test would be based on the *lower* tail area of the distribution of c. The same criterion, but using the upper tail area of

* This is literally true if it is assumed as in (1) that $q_0 < \infty$. In the limiting case when $q_0 = \infty$, if we suppose a *particular* rectangular distribution of semi-range ρ to be the alternative and if r is the radius of a hemispherical shell, then the region defined on the shell will contain an amount of probability density *greater* than any other region of similar size if $\rho < r < \sqrt{N}\rho$. For all other values of r it will contain an amount of probability density *equal* to that of other regions of the same size. The test is thus still most powerful in this limiting case.

c, has been proposed by Thompson (1935) as a test for a single outlier (see also Pearson & Chandra Sekar, 1936). Sections of critical regions for this criterion are shown in Fig. 2 (i). It will be noted that only the lower tail areas of c, b_2 and a define most powerful tests for the alternatives considered. In Fig. 2 the regions are differentiated by drawing in the axes of symmetry only for those sections which correspond to most powerful tests. We should expect c and b_2 to be particularly good criteria, therefore, when the alternative hypothesis was that the distribution showed marked platykurtosis; on the other hand, in testing for marked leptokurtosis a would be expected to be better. Pearson's (1935) sampling experiments (his Table 7) which are shown again in table 4 of the article appearing on p. 318 of this journal support this suggestion (although with so few results no firm conclusions can be drawn). For the rectangular distribution a larger number of significant results is obtained with b_2, whilst for the double exponential a larger number is obtained with a. It does not of course follow that a would become better than b_2 as soon as γ_2 was greater than zero; the change-over might well be at some other point.

In a study of possible tests of kurtosis of this type, and assuming the alternative distribution to be the symmetrical Gram-Charlier, Geary (1947) reached the conclusion that b_2 was the most 'efficient' test for *infinitely large samples* (and correspondingly small departures from normality). He also showed that, over a wide range of values of q, the power of the various possible tests would not be expected to be greatly different. The present research emphasizes that the choice of criterion for samples of finite size must depend on the type of alternative hypothesis which is in mind. The similarity of the critical regions for the various criteria confirm Geary's second conclusion.

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The frequency justification of sequential tests—addendum

By G. A. BARNARD, *Imperial College*

In my paper (1952) on the above topic, at the end of §3 (p. 147) an inversion of limit operations was performed without explicit justification. This has caused difficulty for some readers, and a justification for this process is now appended.

The omission of this proof in the first place was quite deliberate. In the opinion of the writer, statistics is a branch of applied mathematics in which undue attention to pure rigour is out of place. In other branches of applied mathematics, in mechanics, for example, one is expected to make use of experience to fill in what would otherwise be gaps in proofs. If a plane lamina is under consideration, and its surface density is given, one does not usually find the assumption explicitly mentioned that the lamina is a measurable set of points before use is made of the fact that the lamina will have a determinate mass. But in certain quarters great emphasis seems to be laid on just such assumptions in statistics.

It is not that the present writer is opposed to purely mathematical arguments. It is, that such arguments should be kept in their proper place, and that we should not pretend we are doing statistics when we are really doing nothing but pure mathematics.

In §5 of the paper, it is stated that 'Jeffreys has not ... laid down general principles which determine the choice of prior distributions in cases of "ignorance"'. This should read '...which *uniquely* determine the choice...'.

We now give the proof referred to above.

We wish to show that if

$$(a)\ (1/\log b/a)\int_a^b \alpha^*(\sigma, a, b)\, d\sigma/\sigma = \alpha' \quad \text{for all } 0<a<b<\infty$$

and

$$(b)\ \text{as } a \to 0 \text{ and } b \to \infty,\ \alpha^*(\sigma, a, b) \to \alpha,$$

then $\alpha = \alpha'$.

We notice first that $\alpha''(\sigma, a, b) = \alpha''(c\sigma, a/c, b/c)$ for any positive c. For if we make the changes $X \to cX$, $a \to a/c$, $b \to b/c$, then $\sigma \to c\sigma$, $s \to cs$, sa, sb, and t remain unaltered, and so $\lambda_n(a, b)$ remains unaltered. It follows that we can write $\alpha''(\sigma, a, b) = \alpha''(\sigma a, \sigma b)$.

We now put $t = \log \sigma$, $l_1 = \log a$, $l_2 = \log b$, in the integral, and put $\alpha''(\sigma a, \sigma b) = f(t + l_1, t + l_2)$. Then (a) becomes

$$(1/(l_2 - l_1)) \int_{l_1}^{l_2} f(t + l_1, t + l_2)\, dt = \alpha' \text{ for all } l_1 < l_2,$$

while (b) gives $f(x, y) \to \alpha$ as $x \to -\infty$ and $y \to +\infty$. The second condition implies that, given any positive ϵ, we can find an A such that

$$|f(x, y) - \alpha| < \epsilon \text{ whenever } x < -A \text{ and } y > A.$$

Now in (a), put $l_1 = -nA$, $l_2 = +nA$, with $n > 1$. Then we have

$$\alpha' = (1/2nA)\left\{\int_{-nA}^{-(n-1)A} f(t - nA, t + nA)\, dt + \int_{-(n-1)A}^{+(n-1)A} f(t - nA, t + nA)\, dt + \int_{(n-1)A}^{nA} f(t - nA, t + nA)\, dt\right\}$$

$$= (1/2nA)\{Af(t_1 - nA, t_1 + nA) + 2(n-1)Af(t_2 - nA, t_2 + nA) + Af(t_3 - nA, t_3 + nA)\},$$

where $\quad -nA < t_1 < -(n-1)A, \quad -(n-1)A < t_2 < (n-1)A \quad \text{and} \quad (n-1)A < t_3 < nA,$

by the mean-value theorem. Now f is a probability, and is therefore bounded, so that as $n \to \infty$, the first and last terms within the braces remain bounded. Hence, by taking n large enough, we can make α' differ by less than ϵ from $f(t_2 - nA, t_2 + nA)$. But $t_2 - nA < -A$ and $t_2 + nA > A$, so that $f(t_2 - nA, t_2 + nA)$ differs by less than ϵ from α. Hence α and α' differ by less than 2ϵ.

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REVIEWS

The Design and Analysis of Experiments. By O. KEMPTHORNE. New York: John Wiley and Sons; London: Chapman and Hall. 1952. Pp. xix + 631. 76*s*.

Until two or three years ago there were few connected and comprehensive accounts of the theory and practice of experimental design. There were the classical monographs by Fisher and Yates and a vast journal literature on subsequent developments. To the statistician outside the field of agricultural experimentation, the ingenuity shown in devising more and more refined and specialized designs was indeed impressive, but the resulting complexity became, it must be admitted, somewhat baffling.

There was clearly a need for a book which would give a unified account of all the designs, their advantages and disadvantages and their relation to general experimental principles, and which would connect all this with modern statistical theory. As is inevitable when a gap in the literature is disclosed, not one but several books were written to fill it. Quite apart from those recent statistical text-books which have given prominence to experimental design (among which that by Bancroft and Anderson is a notable example) at least four books entirely devoted to this subject have appeared in the last four years. (It is regrettable, incidentally, that their titles are virtually identical.) The first, by Mann (*Analysis and Design of Experiments*), concerns itself strictly with the mathematics underlying the analysis of different designs. The second is the well-known volume by Cochran and Cox. The third, by Kempthorne, is the one under review and the fourth, by Quenouille (*The Design and Analysis of Experiment*), only appeared in the United Kingdom in early 1953. It is notable, though in no way surprising, that three of these books are written (or partly written) by statisticians who have worked at Rothamsted.

As might be expected with two books from the same publisher, the books by Cochran and Cox and by Kempthorne differ in approach. Professor Kempthorne says that 'the basic requirements of the user of experimental designs have been satisfied by the book of Cochran and Cox.... The aim of this book has been to give a description of the design of experiments from as broad a view as possible and to relate the subject matter of this field of statistics to the general theory of statistics and to the general problem of experimental inference.' This should not be taken to mean that Cochran and Cox offer no guidance on general principles of design or on the advantages and otherwise of different designs, for they are useful and clear on both; but their book is a manual of experimental design, that by Kempthorne is an advanced text-book. It is the former which will be of greater service to most practical experimentalists; the latter which, by and large, will be of more interest to the theoretical statistician, whether as consultant or teacher. There is something to be said for using the two books in conjunction. Possibly because its approach is more difficult, Kempthorne's text does not always attain the clarity of exposition which distinguishes Cochran and Cox's book, and reference from one to the other often helps. In any case, this new book is not for the beginner, nor is it suitable for superficial and easy reference. It should be looked upon as a comprehensive and advanced text-book and studied with the concentration and patience such a book deserves.

Its comprehensiveness is indeed remarkable and Professor Kempthorne must be congratulated for carrying through so successfully the enormous task of providing a systematic treatment of experimental designs. He begins with six more or less introductory chapters, the first two on experimental principles, the next on elementary statistical notions (the value of this chapter is open to doubt; it is too scanty for those without sufficient background and superfluous for others) and three on least squares and general linear hypothesis theory. Chapters 7 and 8 between them give an admirable treatment of randomization. Chapters 9 and 10 deal with the two basic designs, randomized blocks and latin squares, and are followed by one on plot technique. Chapter 12 is on the sensitivity of experiments, a subject which has had relatively little attention in other texts. Then follows the core of the book, namely, nine chapters dealing with factorial designs. After this come four chapters on lattice designs, two on incomplete block designs, one on the analysis of groups of experiments and a final one, chapter 29, on treatments applied in sequence. The mathematical theory is given for all the designs, together with explanations and numerical examples. Judged by any standard, the book is a fine achievement.

As there will undoubtedly be future editions, it may be worth mentioning one general aspect which could be improved. Of some subjects treated by Professor Kempthorne one has the impression, not so much of a unified treatment, as of a series of notes scattered throughout the volume. The analysis of covariance is a case in point. It is first introduced in § 6·6 as a modification to the general two-way

model; the mathematics for it are given at length, but no explanation is given of its purpose or use. This comes later, first in § 8·7 in the chapter on The Validity of Analyses of Randomized Experiments and then in § 9·9 in the chapter on Randomized Blocks. (The index does not refer the reader to § 8·7). A certain amount of rearrangement on this, as on some other topics, would clarify the presentation. The very first opportunity should also be taken for improving the index and the cross-referencing. The present index is quite inadequate and there is virtually no cross-referencing in the text.

The theory and practice of experimental design has grown so much out of agricultural experimentation that it is inevitable that the discussion and illustrations in this book refer almost wholly to this field. At several points, Professor Kempthorne remarks critically on methods used in the social sciences. It is of course true that experimentation in this field is very much more difficult than in agriculture. But it is not impossible nor are survey techniques as limited in their value as Professor Kempthorne suggests, in passing, in his first chapter. In the last year or two, attempts have been made to use experimental designs in investigations dealing with human populations and something has been learned about the difficulties and possibilities. For the statistician working in the social sciences, one of the most interesting problems is, to what extent experimental techniques, such as those discussed in this book, can be imported into his domain of study. For him, as for his statistical colleagues in other sciences and fields of activity, Professor Kempthorne's book is a most welcome addition to the literature.

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Associated Measurements. By M. H. QUENOUILLE. Pp. x + 242. Butterworth. 35*s*.

The Design and Analysis of Experiment. By M. H. QUENOUILLE. Pp. xiii + 356. Griffin. 36*s*.

To have produced these two books within a comparatively short time is a notable achievement. The books have much in common; they are advanced non-mathematical text-books on statistical methods excellently illustrated with examples, mostly biological, which are handled with great skill and with a clear sense of the practical purpose of the statistical analysis. The main fault is an occasional carelessness of writing.

The discussion is mostly in terms of specific examples and, while the account of these is remarkably clear, the reader without a sound background of theory may find it difficult to apply the more advanced methods to cases slightly different from those considered by Mr Quenouille. For example, the method of least squares, which underlies much of both books, is, so far as I could see, never mentioned and is certainly never explained. It is true that the stage-by-stage fitting of groups of constants in a least-squares model is often most expeditiously done by analysis of covariance, and that this technique is used repeatedly. But a statement in some of the more complicated cases of the form that is assumed for the underlying population would have given a unity which is at present lacking. An excellent point about both books is the care taken to explain the assumptions necessary for the validity of the various methods.

Associated Measurements starts with a very good section on graphical analysis, including an account of non-parametric tests for association. The description here of tests of partial correlation is open to criticism. The author defines a *coefficient of medial correlation* between two variates x and y by dichotomizing the scatter diagram at the medians, finding the proportion, p, of observations in the positive quadrant, and putting

$$\phi_{xy} = 4p - 1. \tag{1}$$

For an infinite population ϕ_{xy} takes values plus and minus unity for perfect correlation, and zero for independence. Also when x and y are independent, the distribution of ϕ_{xy} does not depend on the form of the population, so that a non-parametric test of independence may be obtained. For a bivariate normal population ϕ_{xy} is related to the product-moment correlation coefficient, ρ_{xy}, by W. F. Sheppard's formula

$$\rho_{xy} = \sin\left(\frac{\pi\phi_{xy}}{2}\right). \tag{2}$$

Mr Quenouille next defines a partial medial correlation coefficient by

$$\phi_{xy.z} = \frac{\phi_{xy} - \phi_{xz}\phi_{yz}}{(1-\phi_{xz}^2\phi_{yz}^2)^{\frac{1}{2}}}. \tag{3}$$

No derivation of this formula is given, and although it is used consistently, one suspects that it is a misprint for

$$\phi'_{xy.z} = \frac{\phi_{xy} - \phi_{xz}\phi_{yz}}{(1-\phi^2_{xz})^{\frac{1}{2}}(1-\phi^2_{yz})^{\frac{1}{2}}}, \tag{4}$$

derived by analogy with product-moment correlations. Both (3) and (4) can give quite misleading results. To see this it is enough to consider some particular trivariate normal populations. For example, suppose that $x = y + z$, where y and z are independently distributed in unit normal distributions. Then, conditionally on z, x is perfectly correlated with y. But $\phi_{xy.z} = \phi_{xy} = \frac{1}{2}$ and $\phi'_{xy.z} = 1/\sqrt{3}$, so that the partial ϕ's suggest that the correlation is unchanged, or only slightly improved, by holding z fixed. The situation is more satisfactory if, conditionally on z, x is independent of y. For, in this case, $\phi_{xy.z}$ and $\phi'_{xy.z}$, while not necessarily zero, are always appreciably smaller than ϕ_{xy}. However the vanishing of $\phi_{xy.z}$ and $\phi'_{xy.z}$ does not imply that $\rho_{xy.z}$ is small. For example if $\rho_{xy} = 0{\cdot}696$, $\rho_{xz} = \rho_{yz} = 0{\cdot}891$, then $\rho_{xy.z} = -0{\cdot}476$, although $\phi_{xy.z} = \phi'_{xy.z} = 0$. Also by taking ρ_{xy}, etc. near unity, $\rho_{xy.z}$ may be made arbitrarily near minus one, with $\phi_{xy.z} = \phi'_{xy.z} = 0$. Enough has been said to make it clear that the uncritical use of partial ϕ could be very misleading.

Similar remarks may, incidentally, be made about the coefficient of partial rank correlation given by Prof. M. G. Kendall in his *Rank Correlation Methods*. Although the theory of rankings derived by sampling an underlying normal population is only a very special case of the general theory of ranking, it is worth noting that in such cases Kendall's partial τ may give a wrong idea of the form of the continuous population. The similarity with ϕ arises because $\bar{\tau}$, the expectation of τ in samples from a normal universe, is given by the analogue of (2),

$$\rho = \sin\left(\frac{\pi\bar{\tau}}{2}\right). \tag{5}$$

When the underlying population is normal these difficulties may of course be avoided by transforming ϕ or τ by (2) or (5) before calculating the partial correlation. There are a number of objections to this as a general procedure.

The second section of *Associated Measurements* is called Numerical Analysis and contains a brief account of analysis of variance, analysis of covariance and regression analysis. The next section, on Rapid Estimation and Analysis, includes chapters on the quick estimation of regressions by the coarse grouping of the observations, and on further uses of analysis of covariance. The last three chapters are entitled Analytical Complications, Time Series Analysis and Multivariate Analysis. The chapter on time series deals mainly with useful approximate methods based on correlation coefficients and analysis of covariance. The final chapter on multivariate analysis seems to me one of the best brief introductions to the subject at present available.

The Design and Analysis of Experiment begins with a twenty-page account of the general principles of design. This is so terse that it is unlikely to make much appeal to an experimenter not already fairly familiar with the methods described. For example there is a bald statement that randomization is essential if unbiased estimates and unbiased estimates of error are to be obtained, but there is no attempt to explain in non-statistical language just what the consequences are of failure to randomize.

The following chapters deal with randomized blocks, Latin squares, and with simple and general factorial and split-plot designs. The emphasis tends to be on the appropriate forms of analysis of variance and covariance.

Section B consists of four chapters on incomplete block designs. The main types of confounding and fractional replication are well described. The advice (p. 111) that, except in special cases, the interactions to be confounded should be chosen at random from interactions of a given order, is surprising. Such a randomization is in any case quite pointless, and is dangerous in that it tends to distract attention from the great importance of a careful choice of the interactions to be confounded. The fairly general condemnation of partial confounding (p. 117), because of the complexity of the calculations involved in the analysis, is also open to question.

Sections C and D entitled Long-term Policy and Experimental Complications contain a good deal of interesting material, the discussion of the combination of results from a series of experiments being particularly good.

To sum up, while these books should be read critically, they are very useful and should appeal to anyone wanting a non-mathematical account of advanced statistical methods.

D. R. COX

The Statistics of Bio-assay. By C. I. Bliss. New York: Academic Press, Inc. Pp. iii + 184. $3.50.

Statistical Method in Biological Assay. By D. J. Finney. London: Charles Griffin and Co. Ltd. Pp. xix + 661. 68*s*.

Probit Analysis. By D. J. Finney. Cambridge University Press. 2nd ed., pp. xiv + 318 35*s*.

The statistical theory of bio-assay has been almost entirely developed during the past 20 years and has reached a reasonably stable state, so that the appearance of text-books by two of the foremost workers in this field is extremely welcome. The two general texts cover much the same ground, though Finney's book is by far the more detailed; Bliss's book originally appeared as a single chapter in György's *Vitamin Methods* and is necessarily rather compressed. Both authors deal with parallel line assays with quantitative and quantal responses, slope-ratio assays, the use of covariance and the combination of potency estimates; Bliss has an interesting section on control charts, while Finney includes several topics which do not arise in vitamin work and so fall outside the scope of the shorter book.

Inevitably in so brief a compass, Bliss's book is little more than a collection of recipes—indeed he refers his readers to Finney for all mathematical details. Within this limitation it should prove very valuable. The needs of the practising biologist are always kept in mind, and the author is bold enough to suggest approximate rule-of-thumb techniques for important problems, rather than to decline to commit himself on the grounds that no exact solution is yet available.

Finney's book is substantial to the point of being encyclopaedic, and some of the theory set out is well ahead of practice, at any rate in the assay field. The reader is supposed to be reasonably familiar with the basic statistical techniques, but apart from this, the suggested analytical methods are explained in full detail with an excellent set of worked examples. The underlying logic of bio-assay is discussed at length, and a number of incomplete block designs are given which should prove of great practical use. Four chapters are devoted to quantal assays, and these are noteworthy for a very full discussion of the various alternatives to probit analysis that have been suggested from time to time. The book includes a 40 page Appendix of Tables, many of them new.

The choice between these two books is not an easy one. Statisticians will probably prefer Finney. Biologists may like to keep Bliss's book on the bench (not least because of its handy size), but a study of Finney's book will be necessary for non-routine problems, and may well enable them to improve on their accepted techniques.

A second edition of Finney's *Probit Analysis* is very welcome, the first edition having been unobtainable for some time. Few changes have been made in the original material, but a substantial new chapter on Recent Developments has been added, and some of the tabular matter is new (the table of weighting coefficients now covers natural mortality up to 90%!). Alternatives to probit analysis are still rather inadequately treated (as the author himself points out), but this has been remedied by the very full discussion in the later book. The omission of one or two other topics, such as Tocher's analysis of grouped data and Finney's own work on graphical methods and the adequacy of a single cycle of calculation, seems to be due to a long delay in publication—the preface to the new edition is dated December 1949. The standard of production leaves something to be desired; the new edition has been produced by an offset-litho process on rather thick paper, and the appearance of the page has suffered somewhat. It seems a pity that the price should still be almost double that of the original edition. Nevertheless, *Probit Analysis* remains an essential book for the many biologists and statisticians who work with sigmoid curves.

M. J. R. HEALY

Statistical Theory with Engineering Applications. By A. Hald. New York: Wiley; London: Chapman and Hall. 1952. Pp. xii + 783. 72*s*.

Statistical Tables and Formulas. By A. Hald. New York: Wiley; London: Chapman and Hall. 1952. Pp. 97. 20*s*.

This book has as its aim the provision, in simple and logical form, of those parts of statistical methods which are of use to the engineer in his daily work. Nevertheless, in its 800 pages it covers considerably more ground, and that ground in greater detail, than might be expected from this simple aim. The result has been to produce a text-book which derives from first principles and illustrates with practical examples all the common statistical processes.

The book assumes no previous knowledge, and although advanced mathematical techniques are avoided the treatment and mode of development is mathematical in form. Nearly all the important results derived mathematically are subsequently illustrated practically. The contents include probability, graphs, the fundamental distributions, the analysis of variance, sampling investigations, the analysis of regressions and sequential analysis. It is noteworthy that the various distributions which are studied take up nearly 350 pages, which gives some idea of the thoroughness of the treatment. The many practical examples that are provided to illustrate the theory are drawn almost exclusively from engineering sources but they are put down in such a way that no detailed knowledge of engineering technique is required and the reader could easily substitute his own examples from his particular sphere of applications. The only criticism that could be levelled against the examples is that they are all chosen to illustrate just one point at a time, and one feels that, at any rate towards the end of the book, more complex illustrations should be used bringing in simultaneously a number of points from the theoretical development. As it is, few practical problems that the engineer is likely to meet with will be as simple as those given in the book. Also given in conjunction with each piece of theory are references to original memoirs and papers which describe practical illustrations. The reviewer feels that such references, instead of being scattered, would be better gathered together at the end of each chapter. One other minor point is that far too many equations are numbered, thereby giving them a certain prestige value. For example, in Chapter 18 no less than 157 equations receive numbers and few are referred to again nor are they likely to be quoted.

The particular portion of the book that struck the reviewer as one of the best of its kind was the section on the analysis of variance which is developed extremely logically. The hierarchical and cross-classification types of problem are discussed in some detail from the view-point of both the systematic and the random models with a very lucid explanation of the concept of an interaction. Later the case where there is a mixture of the two models is discussed. The technique of the analysis of variance has great value for the engineer and it is rather a pity that the treatment here is not quite as full as in other parts of the book. Two examples of this may suffice. The analysis of variance is built up on the twin assumptions that the residuals have equal variances and are normally distributed. If the former does not hold then a transformation can sometimes be applied and although this is mentioned on page 427 the full implications are not brought home. The latter form of departure would seem at first sight to indicate that the usual F or z test will not give the true significance level. It has, however, been shown that the non-normality of the residuals does not, in fact, affect the theoretical significance levels very much. Nevertheless if this point is not discussed it is inclined to worry the student. Secondly a short account of the simpler techniques of probit analysis would have been an extremely useful addition to the chapter on linear regression. Perhaps also, in a book of this size, a fuller account of time series might have been expected.

All authors have their own ideas as to the logical order of development of their subject. In this book the discontinuous distributions such as the binomial are dealt with in outline only before the main part of the book on continuous distributions and they are then returned to in more detail later. An alternative scheme, which would appear to have much in its favour, would have been to have dealt with all distributions in this manner and then to devote later chapters tq problems of estimation, confidence limits, testing hypotheses and so on, bringing in the various distributions already discussed.

As a companion to the book a paper-covered set of tables has been issued and there are many references in one volume to the other. These tables are partly statistical, such as those for the normal and t distributions, and those giving confidence limits for the binomial probability, and partly mathematical, such as those for $\log n!$, square roots and reciprocals. All told there are nineteen tables and they cover the work in the book. Viewed in this light they are adequate and are set out in the most useful manner once the reader has got used to the word 'fractile' for significance level. They are, however, not full enough to stand by themselves as a book of statistical tables, especially since the introduction provided is only of very limited value without the book at hand as a guide.

Summarizing, one may classify this book of Hald's as a text-book, pure and simple. There are no discursions into the realms of debatable or controversial issues and hence it is a useful work which can be studied with much profit by a person intent on getting a sound working knowledge of statistics whether engineer or no. At the same time there is a risk that its length may put off all but the keenest student, particularly if he must find the money to buy it himself.

P. G. MOORE

Statistical Decision Functions. By A. WALD. New York: John Wiley and Sons, Inc.; London: Chapman and Hall, Ltd. 1950. Pp. ix + 179. 44*s*.

By the tragic death of Abraham Wald in December 1950 the world of mathematical statistics lost one of its most active and ingenious minds. The present book, published just before his death, shows him to have been at the height of his powers.

To explain what a decision function is, and to introduce concretely the leading ideas defined by Wald, we will consider the following simple example:

An urn contains black balls and white balls, and the proportion p of black balls is unknown. A fixed number n of balls is drawn, at random with replacement, and r of them are observed to be black. We are required to make a guess, p', at the value of p, and if we are wrong we lose an amount of money proportional to $L=(p-p')^2$.

Any rule by means of which we arrive at our guess will be a decision function for this situation. Since n is fixed, the only quantity on which our guess can depend is r, but the value of r need not uniquely determine our guess; we may make p' depend also on the toss of a coin or some other random event. Thus examples of decision functions $d(r)$ in this situation are:

(i) $d(r)=r/n$, (ii) $d(r)=\frac{1}{2}$,

(iii) $d(r)=\lambda(r/n)+\mu(\frac{1}{2})$, for some $\lambda=1-\mu$, $0<\lambda<1$,

(iv) $d(r)=r/n$ with probability $\frac{1}{2}$
$=\frac{1}{2}$ with probability $\frac{1}{2}$.

In cases (i)–(iii) we take $p'=d(r)$, while in case (iv), which is an example of a *randomized* decision function, we toss a coin, and if it gives heads we take $p'=r/n$, while if it gives tails we take $p'=\frac{1}{2}$.

If we could attach, to each decision function d, a definite monetary loss incurred by its use, we would find the problem of choosing between one decision function and another quite simple. We would simply choose that function which minimized the loss. But such a simple solution is impossible, for two reasons. For one thing, for a given value of p the value of r will vary randomly, and so the guess p' will vary, and the loss L will vary along with p'; and for another thing, the distribution of p' will in general depend on p, and, for a given p', the loss L depends on p. Wald disposes of the first difficulty by noting that, for given p, and given d, the distribution of p', and hence that of L, is fixed, so that we may evaluate the mean value of L as a function of p and d. This function Wald calls the risk function, $\rho(p, d)$.

In the four examples we have given the values of ρ are

(i) pq/n, (ii) $\frac{1}{4}-pq$, (iii) $\lambda^2pq/n+\mu^2(\frac{1}{4}-pq)$, (iv) $\frac{1}{8}-pq(1-(1/n))/2$.

Wald then adopts the principle that it is reasonable to act so as to minimize the expected value of the loss, when this can be calculated. This implies that if we have two risk functions, $\rho(p, d_1)$, $\rho(p, d_2)$, the first of which is, for a given value of p, less than the second, then, for this value of p, the first decision function is 'better' than the second, in the sense that it is reasonable to act on the first decision function rather than the second. If the first decision function is sometimes better, and is never worse, than the second, Wald says the first is 'uniformly better' than the second. Taking our decision functions (i)–(iv), we see that (i) is best of all when $p=0$ or $p=1$, since then its risk function is zero; while (ii) is best of all when $p=\frac{1}{2}$; on the other hand, when $p=\frac{1}{2}$, (i) is worst of all, while when $p=0$ or 1, (ii) is worst of all. If we from now on take $\lambda=\sqrt{n}/(1+\sqrt{n})$, the risk function (iii) becomes $\frac{1}{4}(1+\sqrt{n})^2$, which is independent of p and uniformly less than (iv). Thus (iii) is uniformly better than (iv), but it is not uniformly better than (i) or (ii); nor is either (i) or (ii) uniformly better than (iii). Thus, if we confine attention to these four decision functions, the principle of minimizing the mean loss enables us to rule out (iv) from consideration, but leaves us with (i), (ii) and (iii) to choose from. These three are what Wald calls 'admissible' decision functions, because for none of them does there exist another decision function which is uniformly better. (iv) is not admissible. If Mr A uses an admissible decision function to guide his conduct, no one who accepts the principle of minimizing mean loss can say that Mr A is acting unreasonably, since for some possible situations the rule Mr A is using is at least as good as any other.

We can go further if we are prepared to rule out cases where it is impossible to minimize the mean loss—such as would arise in the example we have taken if we allowed n to vary and assumed that further observations cost nothing. In this latter case, for any rule leading to a decision, we could clearly get a better one by taking more observations. But if we rule out such cases, then accepting the principle of minimizing mean loss implies that we need consider as reasonable only admissible

decision functions. Thus the problem of determining the class of all admissible decision functions is seen to be of central importance.

Unfortunately, Wald was unable to formulate simple and general conditions which guarantee that exceptional cases having the same effect as the 'free observations' of the previous paragraph are ruled out. He therefore introduces, instead of the class of all admissible decision functions, the notion of a 'complete class' of decision functions. A class of decision functions is said to be complete if for any decision function d not in the class there is a decision function d' in the class which is uniformly better than d. If we confine our attention to decision functions belonging to a complete class we shall not lose anything thereby, since those we are ignoring are worse than some we are considering. The set of all possible decision functions is itself a complete class; so that the idea of a complete class helps us to choose our decision function only in so far as we make our complete class as small as possible. Wald obtains here what is probably the best possible result, viz. that under very general conditions, the set of all Bayes solutions forms a complete class of decision functions.

By a Bayes solution of a decision problem is meant a solution arrived at by giving the unknown parameter (p in our case) a prior distribution, and then using Bayes's theorem to minimize the mean loss. For example, our decision function (iii) can be arrived at by giving p the prior probability function $Kp^{\gamma-1}q^{\gamma-1}$, with $\gamma=\sqrt{n}/2$. When r black balls out of n have been observed, the posterior distribution of p is then given by the probability function

$$Kp^r q^s p^{\gamma-1}q^{\gamma-1}\Big/\int_0^1 Kp^r q^s p^{\gamma-1}q^{\gamma-1}dp,$$

and to minimize the mean squared error we take the mean of this distribution, $(r+\gamma)/(n+2\gamma)$, which with the given value of γ is the same as (iii). The decision function (ii) is a Bayes solution, since it can be obtained by taking it as *a priori* certain that $p=\frac{1}{2}$. On the other hand, the decision function (iv) cannot be obtained as a Bayes solution since we shall always take the mean of the posterior distribution, and this will always be a definite number, not a random variable. Finally, (i) can be arrived at as a limit of Bayes solutions corresponding to the prior distributions given by K/pq for $\alpha<p<1-\alpha$, as α tends to 0. A solution arrived at in this way, as a limit of Bayes solutions, is called by Wald a Bayes solution in the wide sense. His theorem is, that under very general conditions the class of all Bayes solutions in the wide sense is a complete class.

The simple example we have taken does not indicate in the least the extreme degree of generality which attaches to this result of Wald's. Some idea of this generality can be obtained from the consideration that about half the present book is devoted to the statement of the conditions under which the theorem is true, and to its proof. In our example, the successive observations are statistically independent; this restriction is removed. Again, in our case, the number of observations was fixed in advance; this restriction is removed, and in its place very general assumptions are made to the effect that observations cost money. The possibility is allowed that observations of several different kinds can be made. In our case the problem reduced to the estimation of a single parameter; this restriction is removed, so that we need not even be estimating any finite number of parameters, and, if we are, we need not make point estimates of them. Finally, included in the decision function is not only the final decision to be arrived at, but also which observations are to be made at each stage of the experiment, and when experimentation is to be stopped, etc. It is, in fact, difficult to see in what direction any serious possibility of generalization remains. And it is, of course, precisely in its degree of generality that the value of the theorem lies. Its meaning can be summed up by saying that, if we are prepared to accept the principle of minimizing mean loss, when this is known, then we should use Bayes's theorem to guide us, based on some *a priori* distribution.

Thus we would conjecture that a major result of this work of Wald's will be to reinstate Bayes's theorem to the central place it always should have occupied in the theory of decision functions. It lost this place, probably, because of Fisher's attacks on its use in statistical inference. Fisher was quite right in asserting that Bayes's theorem had no application to the problems of statistical inference with which Fisher, and the majority of statisticians after him, have been concerned. But this was taken on many sides to mean that the theorem had no application to the problems, such as those of sampling inspection, which could rightly be regarded primarily as decision problems rather than inference problems.

Another theme which occurs in the present work is that of the 'minimax' decision function. One of the difficulties that has to be faced by anyone who tries to identify the decision problem and the inference problem is, that we have a feeling that the inference to be drawn from a given set of data in relation to a given set of statistical hypotheses should be unique (though it may be uncertain). And the trouble with Bayes solutions is that there are as many of them as there are prior distributions, in

general (apart from cases, for example, where a uniformly most powerful test exists). So some principle has to be adopted which will pick out from the multiplicity of Bayes solutions one which can in some sense or other be regarded as the 'true' solution. Bayes and Laplace, of course, did this by their famous 'equal distribution of ignorance' postulate. Now Wald suggests, instead, the principle of 'minimizing the maximum loss'. This means choosing that decision function for which the maximum of the risk function is least. In the example we have taken, this means using the decision rule (iii). For we have already seen that this rule is a Bayes rule, and hence admissible. This means that no other rule can have a risk function which is never greater than the risk function for this rule. But since the risk function for rule (iii) is a constant, this means that any other risk function must have its maximum value greater than or equal to this constant. In other words, rule (iii) minimizes the maximum loss.

Minimax rules have useful properties, perhaps the most useful being, as in our example, that the risk function for such rules is usually constant. But the principle of adopting the minimax rule cannot be regarded as any less arbitrary than Bayes's and Laplace's principle of the equal distribution of the prior probability. If, in a long series of repetitions of the problem we have considered, *in fact* p has a distribution, concentrated near the ends of its range, then decision rule (i) will *in fact* be the best of those considered; and if *in fact* there is a concentration towards $\frac{1}{2}$, then perhaps (iii), or even (ii), will be better than (i). The prior distribution really is a feature of the problem, and without knowing it we *cannot* obtain a unique solution, which has any absolute title to being called the 'best'.

From the point of view of statisticians mainly concerned with problems of inductive inference, perhaps the most useful aspect of Wald's main result is, that in so far as every admissible rule is a Bayes rule, since the data enter into the posterior probability only via the likelihood function, this represents another way of seeing that the likelihood function 'contains all the information in the data', and hence that the problem of inference reduces to the problem of describing the likelihood function.

G. A. BARNARD

PUBLICATIONS OF U.S. DEPARTMENT OF COMMERCE, NATIONAL BUREAU OF STANDARDS

(i) Table of arctangents of rational numbers. Applied Mathematics Series 11. 1951. Pp. 105. Price \$1.50.

This table gives arctangents for rational arguments m/n, i.e. it provides $\tan^{-1}(m/n)$ to 12 decimals for all pairs of integers m, n with $0<m<n\leqslant 100$. Also shown are $\cot^{-1}(m/n) = \frac{1}{2}\pi - \tan^{-1}(m/n)$ (to 12 decimals) and the sum of squares m^2+n^2.

The main applications of the table are:

(*a*) The conversion of cartesian co-ordinates n, m to polars $\phi = \tan^{-1}(m/n)$ and $r = \sqrt{(m^2+n^2)}$ for which an auxiliary table of square roots would be required.

(*b*) The evaluation of natural logarithms for complex arguments from

$$\log_e(n+im) = \tfrac{1}{2}\log_e(m^2+n^2) + i\tan^{-1}(m/n)$$

for which an auxiliary table of natural logarithms would be required.

(*c*) The evaluation of the Γ-function for complex arguments from the recurrence

$$\log_e\Gamma(x+iy+1) = \tfrac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}(y/x) + \log_e\Gamma(x+iy).$$

The latter property may make the table useful in statistical work in connection with moment generating functions of Γ-variables or Laguerre series.

Auxiliary tables provide formulae for the 'reduction' of arctangents in the forms

$$\tan^{-1}(m/n) = \Sigma f_j \tan^{-1} r_j \quad \text{(Table 1, col. 3)},$$
$$\tan^{-1} m \quad = \Sigma f_j \tan^{-1} m_j \quad \text{(Table 2)},$$

by providing the integers f_j, r_j and $m_j<m$. The table is provided with a short Introduction prepared by J. Todd and giving theoretical foundations and details of interpolation.

(ii) Tables of the exponential function e^x. Applied Mathematics Series 14 (supersedes MT 2). 1951. Pp. 537. Price \$3.25.

This table constitutes a revised edition of the second volume of the series of mathematical tables published by the Work Projects Administration of the City of New York and was originally published in 1939. It represents the most extensive table of the exponential function available if one takes into consideration both the decimal accuracy in e^x and the fineness of the tabular interval in x.

It provides the following functions:

e^x to 18 decimals for	$x = 0{\cdot}0000$	$(0{\cdot}0001)$	$1{\cdot}0000$
to 15 decimals for	$1{\cdot}0000$	$(0{\cdot}0001)$	$2{\cdot}5000$
to 15 decimals for	$2{\cdot}500$	$(0{\cdot}001)$	$5{\cdot}000$
to 15 decimals for	$5{\cdot}00$	$(0{\cdot}01)$	$10{\cdot}00$
e^{-x} to 18 decimals for	$0{\cdot}0000$	$(0{\cdot}0001)$	$2{\cdot}5000$
e^x and e^{-x} to 18 decimals for	0	(10^{-6})	10^{-4}
to 19 significant figures for	1	(1)	100

and auxiliary tables.

It should be noted that e^x and e^{-x} are given in separate tables and, if both required for the same argument, two separate pages have to be consulted. A common argument for $x \leqslant 2{\cdot}5$ would have been preferable.

(iii) A guide to tables of the normal probability integral. Applied Mathematics Series 21. 1952. Pp. 16. Price 15c.

This issue of the series provides, in Part I, a brief index of tables of the normal probability integral. In addition to the reference, information is provided on the type of area tabulated (i.e. whether single tail, two tail, etc.) the decimal accuracy, the tabular intervals and auxiliary quantities (e.g. differences) provided for interpolation. Lay-out and arrangement of this index are therefore similar to those adopted in *An Index of Mathematical Tables* by Fletcher, Miller & Rosenhead (Scientific Computing Service Ltd., London and McGraw-Hill Book Co., New York, 1946), although the present list of specialized tables is more extensive.

Part II gives the corresponding index for tables of the inverse function (normal deviates).

Part III gives principles of interpolation applicable to these tables in general and Part IV a summary of formulae.

(iv) Tables of the Bessel functions $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$. Applied Mathematics Series 25 (supersedes A.M.S.I.). 1952. Pp. 60. Price 40c.

This table represents a subtabulation of the Bessel functions Y_0, Y_1; K_0, K_1 given at a rather wider interval in vol. VI of the mathematical tables series issued by the British Association for the Advancement of Science; certain recalculations were made for small x. The functions tabled are

$$Y_0(x) \text{ and } Y_1(x) \quad \text{for} \quad x = 0{\cdot}0001\,(0{\cdot}0001)\,0{\cdot}050\,(0{\cdot}001)\,1{\cdot}000$$
$$K_0(x) \text{ and } K_1(x) \quad \text{for} \quad x = 0{\cdot}0001\,(0{\cdot}0001)\,(0{\cdot}033)\,(0{\cdot}001)\,1{\cdot}000$$

to the same (varying) decimal accuracy as in the British Association tables with an additional interpolation tolerance of 1–2 units. For small x auxiliary functions are tabled which enable the Bessel functions to be evaluated with the help of log tables. An Introduction by A. N. Lowan gives basic formulae and methods of interpolation mainly based on the tabled differences.

H. O. HARTLEY.

Additional Note by Editor. Since these reviews were written an additional table in this series has been received, which is a re-issue of Table 16 of the N.B.S. Mathematical Tables Project.

(v) Table of arctan x. Applied Mathematics Series 26. 1952. Pp. 188. Price $1.75.

The integral

$$\text{Arctan}\, x = \int_0^x \frac{du}{1+u^2}$$

is tabled to 12 decimal places for the following ranges and argument intervals:

$$x = 0{\cdot}000\,(0{\cdot}001)\,7{\cdot}000\,(0{\cdot}01)\,50{\cdot}00\,(0{\cdot}1)\,300{\cdot}0\,(1)\,2000\,(10)\,10{,}000.$$

Second differences are given throughout.

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